TOPOLOGIES ON MARKED SCHEMES

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Abstract. In this article we define and study the *v-Zariski* and *v-étale* topologies on the category of *marked schemes*. This category was constructed in [D'A25] to define *edged crystalline cohomology.*

Contents

1. The v-Zariski and v-étale topologies

We study the category of *marked schemes*, introduced in [D'A25, Def. 2.2.3], denoted by $\underline{\mathrm{Sch}}$.

1.1. Valuations on marked rings.

Definition 1.1.1. A *multiplicative valuation* on a ring A is a map $|.|_v : A \to \Gamma \cup \{0\}$, where $(\Gamma, \times)$ is a totally ordered abelian group such that the following conditions are satisfied.

(1) $|0|_v = 0$ and $|1|_v = 1$.
(2) $|xy|_v = |x|_v |y|_v$ for every $x, y \in A$.
(3) $|x + y|_v \leq \max\{|x|_v, |y|_v\}$ for every $x, y \in A$.

The *kernel* of $|.|_v$, denoted by $\mathfrak{p}_v$, is the preimage of 0. If $|.|_v$ is a valuation of A we write K_v for the fraction field of $A/\mathfrak{p}_v$ and by R_v the valuation subring of K_v of elements $x \in K_v$ such that $|x|_v \leq 1$. We say that two valuations $|.|_v, |.|_w$ are *equivalent*, if $\mathfrak{p}_v = \mathfrak{p}_w$ and $R_v = R_w$. If A has a marking $h_{\underline{A}}$, then it induces a simple marking on every ring R_v.

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Definition 1.1.2. We write by $\mathrm{Spv}(A)$ the set of equivalence classes of valuations of A and if $\varphi : B \to A$ is a ring homomorphism, we denote by $\mathrm{Spa}(A, B)$ the subset of $\mathrm{Spv}(A)$ of those valuations $|.|_v$ such that $|\varphi(B)|_v \leq 1$. This construction naturally globalise to a morphism $\varphi : X \to Y$ of schemes, and we write $\mathrm{Spa}(X, Y)$ for the set of valuations bounded over Y. If $\underline{X}$ is a marked scheme, then we define $\mathrm{Spa}(\underline{X}) := \mathrm{Spa}(X^\iota, X)$. We also write $\mathrm{Spa}(\underline{A})$ for $\mathrm{Spa}(\mathrm{Spec}(\underline{A}))$.

1.2. The v-Zariski topology.

Definition 1.2.1. An *open ι-immersion* $\underline{Y} \to \underline{X}$ of marked schemes is a morphism such that $Y^\iota \to X^\iota$ is an open immersion.

Lemma 1.2.2. *Let $\varphi : \underline{A} \to \underline{B}$ be a morphism of principal marked rings. The induced morphism $\mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A})$ is an open ι-immersion if and only if there exist $f_1, \ldots, f_n \in A$ such that $\varphi(f_1), \ldots, \varphi(f_n)$ generate the unit ideal in B^ι and such that $\varphi : A^\iota_{f_i} \to B^\iota_{\varphi(f_i)}$ is an isomorphism for every i.*

Definition 1.2.3. A *v-Zariski covering* of $\underline{X}$ is the datum of a set of ι-immersions of marked schemes $\{\varphi_i \colon \underline{Y}_i \to \underline{X}\}_{i \in I}$ such that the following conditions are satisfied.

(1) $\{\varphi_i^\iota \colon Y_i^\iota \to X^\iota\}_{i \in I}$ is a Zariski covering
(2) For every open $\underline{U} \subseteq \underline{X}$ with $\underline{U}$ affine, there exists a finite set J, a map $\mathbf{i} \colon J \to I$, and affine opens $\underline{V}_j \subseteq \underline{Y}_{\mathbf{i}(j)}$ such that $\bigcup_j \varphi_j(\underline{V}_j(\underline{R})) = \underline{U}(\underline{R})$ for every marked valuation ring $\underline{R}$.

Note that Condition (2) is a variant of the following condition.

(2') $\bigcup_i \varphi_i(\underline{Y}_i(\underline{R})) = \underline{X}(\underline{R})$ for every marked valuation ring $\underline{R}$.

The difference, is an additional finiteness assumption that is simply the analogue of the finiteness assumption for fpqc coverings. Note that Condition (2) is minimally stronger than Condition (2') as explained by the following example.

Example 1.2.4. Let R be a DVR with uniformiser π, fraction field K, and residue field k. Consider the ring $A := \prod_{n=0}^\infty R$ and write $\pi^{(n)} \in A$ for the element which is π on the n-th entry and 1 otherwise, $\pi \in A$ for the image of $\pi \in R$ via the diagonal embedding $R \to A$, $S_i \subseteq A$ for the multiplicative subset of A generated by $\pi^{(i)}, \pi^{(i+1)}, \cdots$, and $B_i := S_i^{-1}A$. Write $\underline{X} := \mathrm{Spec}(A, \pi^{-1})$ and $\underline{Y}_i := \mathrm{Spec}(B_i, \pi^{-1})$. Since $A_\pi = (B_i)_\pi$ for every $i \geq 0$, we have that the morphisms $Y_i^\iota \to X^\iota$ are all isomorphisms, so that (1) is satisfied. Note also that this implies that $\underline{Y}_i(\underline{K}) = \underline{X}(\underline{K})$ for every $i \geq 0$, where $\underline{K} := (K, 1)$. We want to show now that the family satisfies (2') as well but does not satisfy (2).

We have that both A and B_i have Krull dimension 1 and the minimal prime ideals of A are in bijection with the set of ultrafilters of $\mathbb{N}$. If $\mathfrak{p}$ is a minimal prime ideal of A associated to a non-principal ultrafilter, then $A/\mathfrak{p} \to B_i/\mathfrak{p}B_i$ is an isomorphism because $A/(e_j)_{j\geq 0} = B_i/(e_j)_{j \geq 0}$. On the other hand, if $\mathfrak{p}_j$ is the kernel of the j-th projection, then $A/\mathfrak{p}_j \to B_i/\mathfrak{p}_j B_i$ is an isomorphism for $j < i$ and the embedding $R \hookrightarrow K$ for $j \geq i$. Thanks to this observations, we deduce that if $\underline{R} := (R, \pi^{-1})$, then $\bigcup_{0 \leq i} \underline{Y}_i(\underline{R}) = \underline{X}(\underline{R})$, but for each $M \geq 0$, we have that $\bigcup_{0 \leq i \leq M} \underline{Y}_i(\underline{R}) \subsetneq \underline{X}(\underline{R})$.

Remark 1.2.5. For Noetherian schemes the v-Zariski topology can be compared with the $\underline{\mathrm{M}}\mathrm{Zar}$ topology in [KM21].

Lemma 1.2.6. *If $\{\varphi_i\colon \underline{Y}_i \to \underline{X}\}_{i\in I}$ is a finite family of morphisms of principal marked schemes such that Condition (1) and (2') are satisfied, then it is a v-Zariski covering.*

Definition 1.2.7. A *big v-Zariski site*, denoted by $(\underline{\mathrm{Sch}})_{\mathrm{vZar}}$, is any site defined as in [Stacks, Tag 020S]. We also denote by $(\underline{\mathrm{Sch}}/\underline{S})_{\mathrm{vZar}}$ the localisation of $(\underline{\mathrm{Sch}})_{\mathrm{vZar}}$ with respect to an $\underline{S} \in \underline{\mathrm{Sch}}$ and by $(\underline{S})_{\mathrm{vZar}}$ the small v-Zariski site of $\underline{S}$.

Lemma 1.2.8. *Every Zariski covering $\{\underline{U}_i \to \underline{X}\}_{i\in I}$ is a v-Zariski covering.*

Proof. We may assume that X is affine, so that there exists a subcovering $\{\underline{U}_j \to \underline{X}\}_{i\in J}$ with $J \subseteq I$ finite. After this reduction the result follows from the fact that if $\underline{R}$ is a marked valuation ring, every R-point of X defines an R-point of U_j for some J. □

Lemma 1.2.9. *Every marked scheme $\underline{X}$ admits a v-Zariski covering of principal affine simply marked scheme.*

Proof. Thanks to Lemma 1.2.8 we may assume that $\underline{X} = \mathrm{Spec}(A, f_1^{-1}, \dots, f_n^{-1})$ is principal affine. We consider the subring $B_1 \subseteq A_{f_n}$ (resp. $B_2 \subseteq A_{f_n}$) generated by the image of A and $\frac{f_{n-1}}{f_n}$ (resp. $\frac{f_n}{f_{n-1}}$) endowed with the marking $\{f_\ell\}_{\ell\in(L\setminus\{n\})}$ (resp. $\{f_\ell\}_{\ell\in(L\setminus\{n-1\})}$). We have that $\{\mathrm{Spec}(\underline{B}_i) \to \mathrm{Spec}(\underline{A})\}_{i\in\{1,2\}}$ is a v-Zariski covering of $\mathrm{Spec}(\underline{A})$. The result then follows by induction on n. □

Lemma 1.2.10. *If (A, I_A^{-1}) is a simply marked ring and I_A is generated by a set $\{f_\ell\}_{\ell\in L} \subseteq I_A$, then $\{\mathrm{Spec}(A, f_\ell^{-1}) \to \mathrm{Spec}(A, I_A^{-1})\}_{\ell\in L}$ is a v-Zariski covering.*

Proof. To prove that $\{\mathrm{Spec}(A, f_\ell^{-1})^\iota \to \mathrm{Spec}(A, I_A^{-1})^\iota\}_{\ell\in L}$ is a Zariski covering it is enough to note that, by the assumption, for every prime ideal $\mathfrak{p}$ which does not contain I_A there exists an f_i which is not in $\mathfrak{p}$. By Lemma 1.2.9, it is enough to prove the result after base change to $\varphi : (A, I_A^{-1}) \to (B, g^{-1})$. Thus we have to show that $\{\mathrm{Spec}(B, \varphi(f_\ell)^{-1}, g^{-1}) \to \mathrm{Spec}(B, g^{-1})\}_{\ell\in L}$ is a v-Zariski covering. For this purpose, we note that there exists a finite subset $L' \subseteq L$ such $g \in (\varphi(f_\ell))_{\ell\in L'}$. Since for every valuation $|.|_v \in \mathrm{Spa}(B, g^{-1})$, there exists $\ell \in L'$ such that $|g|_v \leq |f_\ell|_v$, we deduce that $\{\mathrm{Spec}(B, \varphi(f_\ell)^{-1}, g^{-1}) \to \mathrm{Spec}(B, g^{-1})\}_{\ell\in L'}$ satisfies Condition (2). This concludes the proof. □

Definition 1.2.11. A *big v-Zariski site*, denoted by $(\underline{\mathrm{Sch}})_{\mathrm{vZar}}$, is any site defined as in [Stacks, Def. 020S] using v-Zariski coverings. We also denote by $(\underline{\mathrm{Sch}}/\underline{S})_{\mathrm{vZar}}$ the localisation of $(\underline{\mathrm{Sch}})_{\mathrm{vZar}}$ with respect to a marked scheme $\underline{S} \in \underline{\mathrm{Sch}}$ and by $(\underline{S})_{\mathrm{vZar}}$ the subcategory of ι-open immersions $\underline{T} \to \underline{S}$.

1.3. **Structural sheaf and Serre vanishing.** While the presheaf $\mathcal{O}^\iota$ over $(\underline{\mathrm{Sch}}/\underline{S})_{\mathrm{vZar}}$ which sends $\underline{X} \mapsto \mathcal{O}(X^\iota)$ is clearly a sheaf, the presheaf $\mathcal{O}$ which sends $\underline{X} \mapsto \mathcal{O}(X)$ is not a sheaf. The marked ring epimorphism

$$(\mathbb{Z}[t^2, t^3], t^{-1}) \to (\mathbb{Z}[t], t^{-1})$$

induces a v-Zariski covering of $\mathrm{Spec}(\mathbb{Z}[t], t^{-1})$ with trivial Čech nerve. Therefore, the value of a v-Zariski sheaf on $\mathrm{Spec}(\mathbb{Z}[t], t^{-1})$ and $\mathrm{Spec}(\mathbb{Z}[t^2, t^3], t^{-1})$ is the same. We denote by $\mathcal{O}^+$ the sheafification of $\mathcal{O}$ with respect to the v-Zariski topology. Since $\mathcal{O}(\underline{X}) \to \mathcal{O}^\iota(\underline{X})$ is injective, $\mathcal{O}$ is a separated presheaf.

Definition 1.3.1. The *v-refinement* of $\underline{A}$ is the marked ring $\underline{A}^+$ with A^+ the integral closure of $A \to A^\iota$ and the marking is induced by the marking of $\underline{A}$.

Definition 1.3.2. We say that a morphism of marked scheme $\underline{Y} \to \underline{X}$ satisfies the *existence part of the marked valuative criterion*, if for every marked valuation ring $\underline{R}$ with fraction field K and every solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \underline{Y} \arrow[d] \\ \mathrm{Spec}(\underline{R}) \arrow[r] \arrow[ur, dotted] & \underline{X}, \end{tikzcd}$$

the dotted arrow exists.

Lemma 1.3.3. *Let $\underline{A}$ be a principal marked ring with slicing element f. If $\underline{A}^+$ is the v-refinement of $\underline{A}$, then the square*

$$\begin{tikzcd} A^+ \arrow[r] \arrow[d] & \prod_{v \in \mathrm{Spa}(\underline{A})} R_v \arrow[d] \\ A_f \arrow[r] & \prod_{v \in \mathrm{Spa}(\underline{A})} K_v \end{tikzcd}$$

is cartesian.

Proof. Write D for the fibre product of A_f and $B := \prod_{v \in \mathrm{Spa}(\underline{A})} R_v$ over $C := \prod_{v \in \mathrm{Spa}(\underline{A})} K_v$. Since valuation rings are integrally closed, there is a natural morphism $A^+ \to D$. In addition, since $B \to C$ is injective, we deduce that $D \subseteq A_f$. It remains to prove that $A \to D$ is integral.

We follow the proof of Tag 01WM and Tag 01KE of [Stacks]. For an element $g \in D$ we write $J \subseteq A[t]$ for the kernel of the morphism $A[t] \to D_g$ which sends t to g^{-1}. We have that g is integral over A if and only if $1 \in J + (t)$. In turn, to check the last condition it is enough to prove that $\varphi : \mathrm{Spec}(D_g) \to \mathrm{Spec}(A[t]/J)$ is surjective. Note that φ is an isomorphism outside $V(f)$ so that every prime $\mathfrak{p} \subseteq A[t]/J$ which does not contain f is in the set-theoretic image of φ. In particular, since f is a nonzerodivisor in $A[t]/J$, every minimal prime of $A[t]/J$ is in the image. It remains to show that if $\mathfrak{p} \subseteq \mathfrak{q}$ are prime ideals in $A[t]/J$ with $f \notin \mathfrak{p}$ and $\mathfrak{p}$ is in the image of φ, then the same is true for $\mathfrak{q}$. Arguing as in [Stacks, Tag 01KE], this follows from the fact that $\mathrm{Spec}(D_g, f^{-1}) \to \mathrm{Spec}(A[t]/J, f^{-1})$ satisfies the existence part of the marked valuative criterion. □

Lemma 1.3.4. $\mathrm{Spec}(\underline{A}^+) \to \mathrm{Spec}(\underline{A})$ *is a v-Zariski covering.*

Proof. The natural map $\mathrm{Spa}(\underline{A}^+) \to \mathrm{Spa}(\underline{A})$ is a bijection and the valuation ring associated to a multiplicative valuation $v \in \mathrm{Spa}(\underline{A}^+)$ is canonically isomorphic to the one associated to the image in $\mathrm{Spa}(\underline{A})$. □

Corollary 1.3.5. *Every marked scheme admits a v-Zariski covering of principal v-refined affine simply marked schemes.*

Proof. By Lemma 1.2.9 every marked scheme admits a v-Zariski covering of principal affine marked schemes and by Lemma 1.3.4 every principal affine marked scheme admits a v-refined v-Zariski covering. □

Definition 1.3.6. Let $\underline{X} = \mathrm{Spec}(\underline{A})$ be an affine marked scheme. We say that $\{\mathrm{Spec}(\underline{B}_i) \to \mathrm{Spec}(\underline{A})\}_{1 \leq i \leq n}$ is a *standard v-Zariski covering* if all the $\underline{B}_i$ are principal v-refined marked rings.

Lemma 1.3.7. *Let $\{\mathrm{Spec}(\underline{B}_i) \to \mathrm{Spec}(\underline{A})\}_{1\leq i\leq n}$ be a v-Zariski covering of affine principal marked schemes. The sequence*

$$0 \to A^+ \to \prod_i (B_i)^+ \to \prod_{i,j} (C_{i,j})^+$$

is exact, where $\underline{C}_{i,j} \coloneqq \underline{B}_i \otimes \underline{B}_j$.

Lemma 1.3.8. *If $\underline{X} = \mathrm{Spec}(\underline{A})$ is a principal v-refined affine marked scheme, then $\mathcal{O}^+(\underline{X}) = A$.*

Proof. By the previous discussion, to compute $\mathcal{O}^+$ it is enough to sheafify $\mathcal{O}$. Since $\mathcal{O}$ is separated, we deduce that

$$\mathcal{O}^+(\underline{X}) = \varinjlim_{\underline{\mathcal{U}}} \check{H}^0(\underline{\mathcal{U}}, \mathcal{O})$$

where the colimit runs over the v-Zariski coverings of $\underline{X}$. By Corollary 1.3.5, it is enough to prove that for every v-refined covering $\{\mathrm{Spec}(\underline{B}_i) \to \mathrm{Spec}(\underline{A})\}_{1\leq i\leq n}$, the sequence

$$0 \to A \to \prod_i B_i \to \prod_{i,j} C_{i,j}$$

is exact. Since $C_{i,j} \subseteq C^+_{i,j}$, Lemma 1.3.7 yields the desired result. ☐

Lemma 1.3.9. *If $f : \underline{Y} \to \underline{X}$ be a separated morphism of marked schemes such that $Y^\iota \to X^\iota$ is an isomorphism, then the diagonal closed immersion $\underline{Y} \hookrightarrow \underline{Y} \times_X \cdots \times_X \underline{Y}$ into the n-fold fibre product over X is a v-Zariski covering for every $n \geq 1$. In addition, if $\underline{Y} \to \underline{X}$ is a v-Zariski covering and $\mathcal{F}$ is a v-Zariski sheaf on $\underline{X}$, then $R\Gamma_{\mathrm{vZar}}(\underline{X}, \mathcal{F}) = R\Gamma_{\mathrm{vZar}}(\underline{Y}, f^*\mathcal{F})$.*

Corollary 1.3.10. *If $\underline{Y} \to \underline{X}$ and $\{\underline{U}_i \to \underline{Y}\}_{i\in I}$ are v-Zariski covering of qcqs marked schemes and $\underline{Y} \to \underline{X}$ is separated, then the v-refined hypercovering associated to $\{\underline{U}_i \to \underline{Y}\}_{i\in I}$ is canonically isomorphic to the one of $\{\underline{U}_i \to \underline{X}\}_{i\in I}$.*

Theorem 1.3.11. *Let $\underline{X}$ be an affine principal marked scheme such that X is smooth over a field k. The cohomology groups $H^i_{\mathrm{vZar}}(\underline{X}, \mathcal{O}^+)$ vanish for $i > 0$.*

Proof. First note that if $\underline{Y} \to \underline{X}$ is a modification, $\mathcal{U} = \{\underline{U}_i \to \underline{Y}\}_{i\in I}$ is a covering of $\underline{Y}$, and $\mathcal{U}_X = \{\underline{U}_i \to \underline{X}\}_{i\in I}$ is the induced covering of $\underline{X}$, then by Lemma 1.3.9 we have that $\check{H}^\bullet(\mathcal{U}, \mathcal{O}^+) = \check{H}^\bullet(\mathcal{U}_X, \mathcal{O}^+)$. Therefore, looking Zariski locally on X and after taking macaulayfication, it is enough to show that for every projective modification $\underline{Y} \to \underline{X}$ with Y Cohen–Macaulay and every Zariski covering $\mathcal{U} = \{\underline{U}_\ell \to \underline{Y}\}_{\ell\in L}$ and every class in $\check{H}^\bullet_{\mathrm{vZar}}(\mathcal{U}, \mathcal{O}^+)$, there exists a Zariski refinement $\mathcal{V} = \{\underline{V}_m \to \underline{Y}\}_{m\in M}$ which kills the class. This follows from Kovacs' vanishing. ☐

1.4. **Comparison with the Zariski cohomology.**

If $\underline{S}$ is any marked scheme, we have a functor

$$v \colon (\underline{\mathrm{Sch}}/\underline{S})_{\mathrm{vZar}} \to (\mathrm{Sch}/S)_{\mathrm{Zar}}$$

which forgets the marking. This functor admits as a right adjoint the functor

$$u_{1/\underline{S}} \colon (\mathrm{Sch}/S)_{\mathrm{Zar}} \to (\underline{\mathrm{Sch}}/\underline{S})_{\mathrm{vZar}}$$

which sends $X \to S$ to $\underline{S} \times_{u_1(S)} u_1(X) \to \underline{S}$. The counit of the adjunction is an isomorphism.

Lemma 1.4.1. *The functor v is cocontinuous and $u_{1/\underline{S}}$ is continuous.*

Proof. If $\{U_i \to v(\underline{X})\}$ is a covering in $(\mathrm{Sch}/S)_{\mathrm{Zar}}$, then $\{u_{1,\underline{X}}(U_i) \to \underline{X}\}$ is a covering in $(\underline{\mathrm{Sch}}/\underline{S})_{\mathrm{vZar}}$ by Lemma 1.2.8. We deduce that v is cocontinuous. □

We write

$$\alpha\colon \mathrm{Sh}((\underline{\mathrm{Sch}}/\underline{S})_{\mathrm{vZar}}) \to \mathrm{Sh}((\mathrm{Sch}/S)_{\mathrm{Zar}})$$

for the map of topoi induced by v (viewed as a cocontinuous functor). We have that $\alpha_*(\mathcal{F})(X) = \mathcal{F}(u_{1/\underline{S}}(X))$ and $\alpha^{-1}(\mathcal{G})(\underline{X}) = \mathcal{G}(X)$ for every $\underline{X} \to \underline{S}$. Moreover, the counit $\alpha^{-1}\alpha_* \to \mathrm{id}$ is given by

$$\alpha^{-1}(\alpha_*\mathcal{F})(\underline{X}) = \alpha_*\mathcal{F}(X) = \mathcal{F}(u_{1/\underline{S}}(X)) \to \mathcal{F}(\underline{X})$$

and the unit $\mathrm{id} \to \alpha_*\alpha^{-1}$ is the identity. By [Stacks, Tag 09YX], α^{-1} admits also a left adjoint $\alpha_!$ such that $\alpha_! h_{\underline{X}} = h_X$.

Lemma 1.4.2. *The sheaf $\alpha_*\mathcal{O}^+_{\underline{X}}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.*

Definition 1.4.3. If $\underline{X}$ is a marked scheme we denote by $\underline{X}^+$ the marked scheme with underlying scheme $X^+ := \mathrm{Spec}_X(\alpha_*\mathcal{O}_{\underline{X}})$ and marking induced by the one of $\underline{X}$ via the natural morphism $X^+ \to X$. We say that $\underline{X}$ is *v-refined* if $X^+ \to X$ is an isomorphism.

Lemma 1.4.4. $\mathrm{Sh}((\underline{X}^+)_{\mathrm{vZar}}) = \mathrm{Sh}(\underline{X}_{\mathrm{vZar}})$.

Lemma 1.4.5. *For a marked scheme $\underline{X}$ we have that the natural square*

$$\begin{array}{ccc} (X_{\mathrm{red}})^+ & \longrightarrow & X^+ \\ \downarrow & & \downarrow \\ X_{\mathrm{red}} & \longrightarrow & X. \end{array}$$

is cartesian.

1.5. **Stalks.** If $\underline{R}$ is a marked valuation ring, for every $\underline{x} \in \underline{X}(\underline{R})$ one associates the functor $p_{\underline{x}}\colon \underline{X}_{\mathrm{vZar}} \to \mathrm{Set}$ which sends $\underline{U} \to \underline{X}$ to the set of $\underline{R}$-points over $\underline{x}$. This defines a point of the site $\underline{X}_{\mathrm{vZar}}$ (cf. [Stacks, Tag 00Y5]).

Definition 1.5.1. If $\mathcal{F}$ is a sheaf of $\underline{X}_{\mathrm{vZar}}$, the stalk of $\mathcal{F}$ at $\underline{x}$, denoted by $\mathcal{F}_{\underline{x}}$, is the inverse limit

$$\varinjlim_{\underline{T}\to\underline{U}\to\underline{X}} \mathcal{F}(\underline{U}),$$

where $\underline{T} := \mathrm{Spec}(\underline{R})$ and the composition $\underline{T} \to \underline{U} \to \underline{X}$ is $\underline{x}$.

The following lemma is related to [KM21, Prop. 4.25].

Lemma 1.5.2. *The family of points of $\underline{X}_{\mathrm{vZar}}$ associated to local rings and marked valuation rings is a conservative family.*

Definition 1.5.3. We say that a morphism of marked schemes $f\colon \underline{Y} \to \underline{X}$ is a modification if $f\colon Y \to X$ is proper and finitely presented and $f\colon Y^\iota \to X^\iota$ is an isomorphism.

Proposition 1.5.4. *Let $f\colon \underline{Y} \to \underline{X}$ be a minimal modification. The v-Zariski higher direct image $Rf_*\mathcal{O}^+_{\underline{Y}}$ is quasi-isomorphic to $\mathcal{O}^+_{\underline{X}}$.*

Proof. Stein's factorisation [Stacks, Tag 03H2] to show $f_*\mathcal{O}^+_{\underline{Y}} = \mathcal{O}^+_{\underline{X}}$. □

1.6. **Closed immersions.** Let $i : \underline{Z} \hookrightarrow \underline{X}$ a minimal closed immersion of marked schemes.

Lemma 1.6.1. *The morphism $i^\sharp : \mathcal{O}^+_X \to i_*\mathcal{O}^+_Z$ is surjective.*

Proof. It is enough to check the surjectivity at the level of stalks. Therefore, we may assume that $\underline{X}$ is the spectrum of a marked valuation ring (R, f^{-1}). Let $R \twoheadrightarrow R/I$ the quotient induced by i. We have to show that $R = R^+_f \to (R/I)^+_f$ is surjective. If $\mathfrak{p}$ is the minimal prime of R which contains I, we have that $R/\mathfrak{p}$ is a valuation ring which coincides with $(R/I)_{\mathrm{red}}$. By Lemma 1.4.5, since $(R/\mathfrak{p})^+_f = R/\mathfrak{p}$, we deduce that $(R/I)^+_f = R/I$. □

1.7. v**-refined Kähler differentials.** Let $i : \underline{Z} \hookrightarrow \underline{X}$ be a minimal closed embedding between v-refined marked schemes. The kernel $\mathcal{I}$ of the morphism $\mathcal{O}^+_{\underline{X}} \twoheadrightarrow i_*\mathcal{O}^+_{\underline{Z}}$ is a quasi-coherent sheaf of $\mathcal{O}^+_{\underline{X}}$-modules. In addition, the conormal sheaf $\mathcal{C}_{\underline{Z}/\underline{X}} := \mathcal{I}/\mathcal{I}^2$ is a quasi-coherent sheaf of $\mathcal{O}^+_{\underline{X}}$-modules. For a morphism $\underline{Y} \to \underline{X}$ of marked schemes we write $\Omega^{1,+}_{\underline{Y}/X}$ for $\mathcal{C}_{\underline{Y}^+/(\underline{Y}\times_X\underline{Y})^+}$.

Lemma 1.7.1. *If $f : \underline{Y} \to \underline{X}$ is a morphism of marked S-schemes, then*

$$f^*\Omega^{1,+}_{\underline{X}/S} \to \Omega^{1,+}_{\underline{Y}/S} \to \Omega^{1,+}_{\underline{Y}/X} \to 0$$

is exact.

Remark 1.7.2. Note that if $\underline{Y} \to \underline{X}$ is an admissible blowing up then $\Omega^{1,+}_{\underline{Y}/X} = 0$.

1.8. **The** v**-étale topology.**

Definition 1.8.1. Let $\underline{X}$ be a principal affine marked scheme. A *standard v-étale covering* of $\underline{X}$ is the datum of a finite set of morphisms of principal affine marked schemes $\{\underline{Y}_i \to \underline{X}\}_{i\in I}$ such that the following conditions are satisfied.

(1) $\{Y_i^\iota \to X^\iota\}_{i\in I}$ is an étale covering.
(2) For every marked valuation ring $\underline{R}$ and every $T \in \underline{X}(\underline{R})$ there exists a minimal extension $\underline{R} \subseteq \underline{R}'$, an $i \in I$, and $T' \in \underline{Y}_i(\underline{R}')$ lifting T.

The *v-étale topology* is the topology generated by the v-Zariski topology and the standard v-étale coverings.

1.8.2. Let k is a field of positive characteristic p, let $\underline{A} \subseteq \underline{B}$ be the minimal extension of marked rings $(k[x], x^{-1}) \subseteq (k[x,y]/(y^p - x^{p-1}y - x^{p-1}), x^{-1})$. Consider the induced morphism $f : \underline{Y} \to \underline{X}$ where $\underline{X} := \mathrm{Spec}(\underline{A})$ and $\underline{Y} := \mathrm{Spec}(\underline{B})$. Note that f is a v-étale covering.

Lemma 1.8.3. *The sheaf $\mathcal{O}^+$ does not satisfy v-Zariski cohomological descent with respect to f.*

Proof. Let G be the Galois group of f (seen as a constant group scheme over k) and let $\underline{Y}_\bullet$ be the Čech nerve of $\underline{Y} \to \underline{X}$. We have that

$$\Gamma(\underline{Y}_\bullet, \mathcal{O}^+) \simeq \mathrm{Hom}_G(\mathbb{Z}[G^{\bullet+1}], B)$$

where $\mathbb{Z}[G^{\bullet+1}]$ is the simplicial group associated to the bar resolution of G and B is endowed with the natural G-action. This implies that the spectral sequence

$$E_2^{i,j} := H^i(G, H^j_{\mathrm{vZar}}(\underline{Y}, \mathcal{O}^+))$$

converges to $H^{i+j}_{\mathrm{vZar}}(\underline{Y}_\bullet, \mathcal{O}^+)$, where the action of G on $H^j_{\mathrm{vZar}}(\underline{Y}, \mathcal{O}^+)$ is the one induced by the action on $\underline{Y}$. We have that $H^0_{\mathrm{vZar}}(\underline{Y}, \mathcal{O}^+) = B$ and $H^i_{\mathrm{vZar}}(\underline{Y}, \mathcal{O}^+) = 0$ for $i > 0$, so that $H^i_{\mathrm{vZar}}(\underline{Y}_\bullet, \mathcal{O}^+) =$

$H^i(G,B)$. Since $G = \mathbb{Z}/p$ as abstract groups, if $\sigma \in G$ is the automorphism which sends $y \mapsto y + x$ and $\mathrm{Tr} : B \to B$ is the endomorphism $1 + \sigma + \cdots + \sigma^{p-1}$, then

$$H^{2i+1}_{\mathrm{vZar}}(\underline{Y}_\bullet, \mathcal{O}^+) = B^{\mathrm{Tr}=0}/(\sigma - 1)B.$$

The unit $1 \in B^{\mathrm{Tr}=0}$ is not in the image of $\sigma - 1$, as one can check after reducing modulo x. We deduce that $H^{2i+1}_{\mathrm{vZar}}(\underline{Y}_\bullet, \mathcal{O}^+) \neq 0$ for every $i \geq 0$. □

More in general, we have the following result.

Proposition 1.8.4. *Let R be a valuation ring with fraction field K and let R^{sep} be the integral closure of R in a separable closure of K. For every nonzero element $f \in R$, we have*

$$H^\bullet_{\mathrm{v\acute{e}t}}(\mathrm{Spec}(R, f^{-1}), \mathcal{O}^+) = H^\bullet(K, R^{\mathrm{sep}}),$$

where $H^\bullet(K, R^{\mathrm{sep}})$ denote the Galois cohomology groups of R^{sep} endowed with the natural Galois action.

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