On the p-adic slope filtration

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Cohomology theories in positive characteristic

Elliptic curves

Let p be a prime number, $\mathbb{F}$ an algebraic closure of $\mathbb{F}_p$.

$$E \subseteq \mathbb{P}^2_{\mathbb{F}} \qquad \text{non-singular cubic curve}$$

$$m \colon E \times_{\mathbb{F}} E \to E \qquad \text{algebraic commutative group law}$$

$$E(\mathbb{F}) = \bigcup_{i \geq 1} E(\mathbb{F}_{p^i}) \qquad \text{infinite torsion group.}$$

ℓ-adic Tate module

If $\ell \neq p$ is a prime, $n \geq 1$

$$E(\mathbb{F})[\ell^n] := \ker \left(E(\mathbb{F}) \xrightarrow{\ell^n} E(\mathbb{F}) \right) \simeq (\mathbb{Z}/\ell^n\mathbb{Z})^{\oplus 2}$$

$$T_\ell(E) := \varprojlim_n E(\mathbb{F})[\ell^n] \simeq \mathbb{Z}_\ell^{\oplus 2} \qquad \text{is a good } H_1(E, \mathbb{Z}_\ell).$$

p-torsion

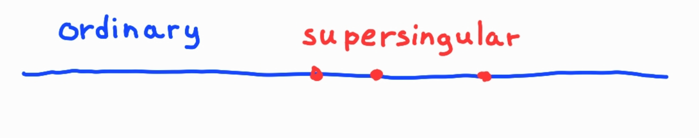
p-adic Tate module

$$E(\mathbb{F})[p^n] \simeq \begin{cases} \mathbb{Z}/p^n\mathbb{Z} & \text{(ordinary)} \\ 0 & \text{(supersingular)} \end{cases}$$

$$\rightsquigarrow \ \mathrm{rk}_{\mathbb{Z}_p}(T_p(E)) < \mathrm{rk}_{\mathbb{Z}_\ell}(T_\ell(E)).$$

If $E = \{y^2 z = x(x-z)(x-\lambda z)\}$ with $\lambda \in \mathbb{F} \setminus \{0,1\}$ and $p \neq 2$, then

$$E(\mathbb{F})[p^n] = 0 \Leftrightarrow \sum_{i=0}^{m} \binom{m}{i}^2 \lambda^i = 0, \quad \text{where} \quad m := \frac{p-1}{2}.$$



Group scheme approach

$$\begin{aligned} E\colon \mathrm{Alg}_{\mathbb{F}} &\to \mathrm{Grp}^{\mathrm{com}} \\ R &\mapsto E(R). \end{aligned}$$

Scheme-theoretic torsion

$$E[p^n] := \ker(E \xrightarrow{p^n} E) \in \mathrm{Fun}(\mathrm{Alg}_{\mathbb{F}}, \mathrm{Grp}^{\mathrm{com}}).$$

Representability

There exists a Hopf algebra A_n over $\mathbb{F}$ of *dimension* p^{2n} such that

$$E[p^n](R) = \mathrm{Hom}_{\mathbb{F}}(A_n, R).$$

$$0 \to E[p^n]^{\mathrm{inf}} \to E[p^n] \to E(\mathbb{F})[p^n] \to 0 \quad \textit{(slope filtration)}$$

with $E[p^n]^{\mathrm{inf}}(\mathbb{F}) = 0$.

Dieudonné module

$$\begin{aligned} E[p^\infty] \colon \mathrm{Alg}_{\mathbb{F}} &\to \mathrm{Grp} \\ R &\mapsto \bigcup_{n\geq 1} E[p^n](R) \end{aligned}$$

is a *p-divisible group (or Barsotti–Tate group)*.

Definition (F-crystal)

Let W be the ring of Witt vectors of $\mathbb{F}$ endowed with the Frobenius lift $\sigma \colon W \to W$. An *F-crystal* over $\mathbb{F}$ is a finitely generated W-module M endowed with a map

$$\Phi_M \colon M^\sigma \to M,$$

which is an isomorphism after inverting p.

Theorem (Dieudonné)

There exists a fully faithful functor

$$\{p\text{-divisible groups}/\mathbb{F}\} \xrightarrow{\mathbb{D}} \{F\text{-crystals}/\mathbb{F}\}.$$

Slopes

Theorem (Dieudonné–Manin)

For every F-crystal (M, Φ_M) over $\mathbb{F}$ there exists $S \subseteq \mathbb{Q}$ such that

$$M[\tfrac{1}{p}] = \bigoplus_{s/r \in S} M_{s/r}[\tfrac{1}{p}]$$

and each $M_{s/r}$ is generated by $v \in M_{s/r}$ such that $(\Phi_M)^r v = p^s v$.

Definition

S is the set of *slopes* of (M, Φ_M).

Example

For $\mathbb{D}(E[p^\infty])$

$$S = \begin{cases} \{0,1\} & E \text{ ordinary,} \\ \{1/2\} & E \text{ supersingular.} \end{cases}$$

Cohomology theories

$$\mathbb{D}(E[p^\infty]) \simeq (W^{\oplus 2}, \Phi_E) \text{ is a good } H_1(E, W)$$

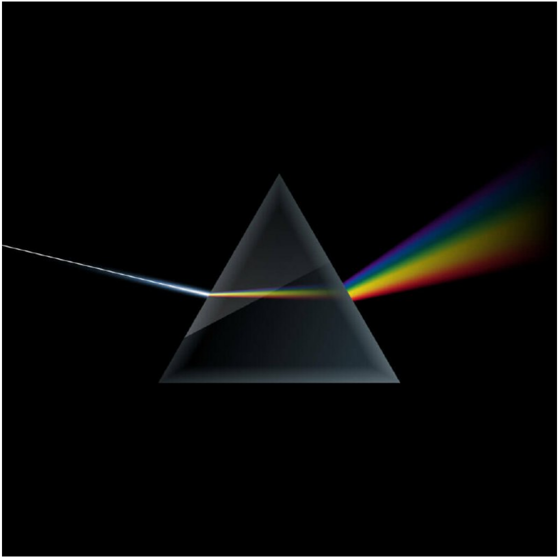
ℓ-adic étale cohomology ($T_\ell(E)^*$)

$$\{\text{Algebraic varieties over } \mathbb{F}\} \to \{\text{Graded finitely generated } \mathbb{Z}_\ell\text{-modules}\}$$
$$X \mapsto H^\bullet_{\text{et}}(X, \mathbb{Z}_\ell)$$

Crystalline cohomology ($\mathbb{D}(E[p^\infty])^*$)

$$\{\text{Smooth projective varieties over } \mathbb{F}\} \to \{\text{Graded } F\text{-crystals over } \mathbb{F}\}$$
$$X \mapsto H^\bullet_{\text{cris}}(X/W)$$

Diffraction of cohomology theories



Local systems

Algebraic varieties in families

$f\colon Y \to X$ smooth proper morphism of smooth connected varieties over $\mathbb{F}$.

ℓ-adic local systems (lisse sheaves)

For $x \in X(\mathbb{F})$,

$$R^i f_{\mathrm{et}*}\mathbb{Q}_\ell : \pi_1^{\mathrm{et}}(X,x) \to \mathrm{GL}(H^i(Y_x,\mathbb{Q}_\ell)) \in \mathrm{LS}_\ell(X).$$

p-adic local systems (F-isocrystals)

$$R^i f_{\mathrm{cris}*}\mathcal{O}_{Y,\mathrm{cris}} \in \text{F-Isoc}(X).$$

Understanding F-isocrystals

If X is affine, it lifts to $\mathfrak{X}/W$ together with a Frobenius lift $F\colon \mathfrak{X} \to \mathfrak{X}$.

F-isocrystals as flat connections

$$\text{F-Isoc}(X) = \left\{ (\mathcal{M}, \nabla, \Phi_{\mathcal{M}}) \;\middle|\; \begin{array}{l} \mathcal{M} \text{ vector bundle over } \mathfrak{X}[\frac{1}{p}], \\ \nabla\colon \mathcal{M} \to \mathcal{M} \otimes \Omega^1_{\mathfrak{X}} \text{ flat connection,} \\ \Phi_{\mathcal{M}}\colon F^*\mathcal{M} \xrightarrow{\sim} \mathcal{M} \text{ compatible with } \nabla. \end{array} \right\}.$$

Example

Let $K := W[\frac{1}{p}]$ and

$$K\langle t \rangle := \left\{ \sum_{i=0}^{\infty} a_i t^i \in K[[t]] \;\middle|\; v_p(a_i) \to \infty \right\}$$

$$\text{F-Isoc}(\mathbb{A}^1_{\mathbb{F}}) = \left\{ (M, D_t, \varphi) \;\middle|\; \begin{array}{l} M \text{ free } K\langle t \rangle\text{-module,} \\ D_t : M \to M \text{ derivation compatible with } \partial_t, \\ \varphi : M^\sigma \xrightarrow{\sim} M \text{ isomorphism such that} \\ D_t \circ \varphi = pt^{p-1} \varphi \circ D_t. \end{array} \right\}.$$

Slope filtration

Theorem (Grothendieck, Katz)

For $(\mathcal{M}, \Phi_{\mathcal{M}}) \in \text{F-Isoc}(X)$, after shrinking X, there is a unique filtration

$$0 = S_0(\mathcal{M}) \subsetneq S_1(\mathcal{M}) \subsetneq \cdots \subsetneq S_m(\mathcal{M}) = \mathcal{M}$$

such that each S_i/S_{i-1} is of pure slope s_i and $s_1 < \cdots < s_m$.

Example (Legendre family)

$f : \mathcal{E} \to X$ Legendre family of elliptic curves, $\mathcal{M} := R^1_{\text{cris}*}\mathcal{O}_{\mathcal{E},\text{cris}}$, and $U := X^{\text{ord}}$,

$$S_1(\mathcal{M}_U) \subsetneq \mathcal{M}_U \quad (\textit{infinitesimal part})$$

is of rank 1. On the other hand, $R^1 f_{\text{et}*}\mathbb{Q}_\ell$ is irreducible. They tell different stories!

Overconvergence

Example (Legendre family)

$\mathcal{M}$ has *regular singularities*, whereas $S_1(\mathcal{M}_U)$ does not have regular singularities. Even when passing to $U' \twoheadrightarrow U$, it does not acquire regular singularities.

Definition (Overconvergent F-isocrystals)

$$\text{F-Isoc}^\dagger(X) \subseteq \text{F-Isoc}(X)$$

is the category of F-isocrystals that acquire regular singularities after passing to a de Jong alteration $X' \twoheadrightarrow X$.

Monodromy groups

Monodromy groups

Definition (ℓ-adic case)

For $\mathcal{V}_\ell \in \mathrm{LS}_\ell(X)$,

$$G(X, \mathcal{V}_\ell) := \text{Zariski closure of the image of the representation}$$

$$\langle \mathcal{V}_\ell \rangle^{\otimes}_{\mathrm{LS}_\ell(X)} = \mathrm{Rep}_{\mathbb{Q}_\ell}(G(X, \mathcal{V}_\ell)).$$

Definition (p-adic case)

For $(\mathcal{M}, \Phi_{\mathcal{M}}) \in \text{F-Isoc}^\dagger(X)$,

$$G(X, \mathcal{M}) \subseteq G^\dagger(X, \mathcal{M})$$

$$\langle \mathcal{M} \rangle^{\otimes}_{\mathrm{Isoc}(X)} = \mathrm{Rep}_K(G(X, \mathcal{M})) \text{ and } \langle \mathcal{M} \rangle^{\otimes}_{\mathrm{Isoc}^\dagger(X)} = \mathrm{Rep}_K(G^\dagger(X, \mathcal{M})).$$

Main results

Theorem (Independence)

If $\mathcal{V}_\ell \in \mathrm{LS}_\ell(X)$ and $(\mathcal{M}, \Phi_{\mathcal{M}}) \in \text{F-Isoc}^\dagger(X)$ come from the cohomology of a smooth proper family, there exists G over $\overline{\mathbb{Q}}$ such that

$$G_{\overline{\mathbb{Q}}_\ell} = G(X, \mathcal{V}_\ell)^\circ_{\overline{\mathbb{Q}}_\ell} \text{ and } G_{\overline{K}} = G^\dagger(X, \mathcal{M})^\circ_{\overline{K}}.$$

Theorem (Crew's parabolicity conjecture)

For $(\mathcal{M}, \Phi_{\mathcal{M}}) \in \text{F-Isoc}^\dagger(X)$ with constant slopes, then

$$G(X, \mathcal{M}) \subseteq G^\dagger(X, \mathcal{M})$$

is the stabiliser of the slope filtration. Moreover, if $\mathcal{M}$ is semi-simple in $\mathrm{Isoc}^\dagger(X)$, then $G(X, \mathcal{M})$ is a parabolic subgroup.

These results form a new bridge between ℓ-adic lisse sheaves and p-divisible groups.

Main results

Applications

- Semi-simplicity of geometric p-adic representations.
- Kedlaya minimal slope conjecture (DA, Tsuzuki).
- Finiteness results for torsion points of abelian varieties (Ambrosi–DA).
- Determination of π_0 of Igusa varieties (van Hoften–Xiao Xiao).
- Characteristic p analogues of the Mumford–Tate and André–Oort conjectures (Jiang).
- **Chai–Oort Hecke orbit conjecture for Shimura varieties** (DA–van Hoften).

Theorem (Local refinement, DA–van Hoften)

If $\mathcal{M} \in \mathrm{Isoc}^{\dagger}(X)$ is semi-simple, then for every $x \in X(\mathbb{F})$

$$G(X^{/x}, \mathcal{M}) \subseteq G(X, \mathcal{M})$$

is the unipotent radical.

Hecke orbit conjecture

The conjecture in the Siegel case

$\mathcal{A}_g$: Moduli space of principally polarised abelian varieties over $\mathbb{F}$ of dimension g.

Definition (Hecke orbit)

For $x \in \mathcal{A}_g(\mathbb{F})$ representing A_x, we define

$$\mathcal{H}(x) := \{y \in \mathcal{A}_g(\mathbb{F}) \mid \exists\ A_y \xrightarrow{\text{isog}} A_x\}.$$

Remark

Over the complex numbers, each Hecke orbit is dense in $\mathcal{A}_g(\mathbb{C})$. This follows from the fact that $\mathrm{Sp}_{2g}(\mathbb{Q})$ is dense in $\mathrm{Sp}_{2g}(\mathbb{R})$.

The conjecture in the Siegel case

Newton stratification

For every isogenous class of p-divisible groups ξ, there exists a locally closed subschemes $\mathcal{N}_\xi \subseteq \mathcal{A}_g$, called *Newton stratum*, parametrising abelian varieties over $\mathbb{F}$ with isogenous p-divisible groups. We have

$$\mathcal{A}_g(\mathbb{F}) = \bigsqcup_{\xi} \mathcal{N}_\xi(\mathbb{F}).$$

Remark

If $x \in \mathcal{N}_\xi(\mathbb{F})$, then $\mathcal{H}(x) \subseteq \mathcal{N}_\xi(\mathbb{F})$.

(Weak) Hecke orbit conjecture for $\mathcal{A}_g$

The Hecke orbit of $x \in \mathcal{A}_g(\mathbb{F})$ is Zariski-dense in the Newton stratum $\mathcal{N}_\xi \subseteq \mathcal{A}_g$ containing it.

Generalised Serre–Tate coordinates

Definition

A *(connected) Dieudonné–Lie W-algebra* is a triple $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$ where $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+})$ is a free F-crystal over $\mathbb{F}$ with slopes in $[0,1)$ and

$$[-,-] : \mathfrak{a}^+ \times \mathfrak{a}^+ \to \mathfrak{a}^+$$

is a Lie bracket compatible with $\varphi_{\mathfrak{a}^+}$. It is *integrable* if the BCH formula is well defined over W.

Given an integrable Dieudonné–Lie W-algebra $\mathfrak{a}^+$ we constructed the following objects.

- $\tilde{\Pi}(\mathfrak{a}^+)$: Perfect formal group scheme $(\hat{\mathbb{A}}^{n,\mathrm{perf}}_{\mathbb{F}}, m_{\mathrm{Lie}})$.
- $\Pi(\mathfrak{a}^+) \subseteq \tilde{\Pi}(\mathfrak{a}^+)$: Profinite semi-perfect group scheme.
- $Z(\mathfrak{a}^+) \coloneqq \tilde{\Pi}(\mathfrak{a}^+)/\Pi(\mathfrak{a}^+)$ fpqc sheaf over $\mathrm{Alg}^{\mathrm{op}}_{\mathbb{F}}$.

Lemma

If $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+})$ admits the slope decomposition, the sheaf $Z(\mathfrak{a}^+)$ is represented by $\hat{\mathbb{A}}^n_{\mathbb{F}}$.

Generalised Serre–Tate coordinates

Special formal subvarieties

Every $\mathcal{N}_\xi$ admits points $x \in \mathcal{N}_\xi(\mathbb{F})$ such that

$$Z(\mathfrak{a}^+) \hookrightarrow \mathcal{N}_\xi^{/x}$$

for explicit $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$.

Serre–Tate coordinates

If $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+})$ is of slope 0, then $[-,-] = 0$ and $Z(\mathfrak{a}^+) \simeq \hat{\mathbb{G}}_m^n$. This construction recovers Serre–Tate coordinates of ordinary abelian varieties.

Theorem (Monodromy bounds, DA–van Hoften)

$$\mathrm{Lie}(G(Z(\mathfrak{a}^+), \mathcal{M})) \subseteq \mathfrak{a}^+[\tfrac{1}{p}].$$