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On the p-adic slope filtration

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This thesis provides a synthesis of our contributions to the study of the slope filtration and the monodromy groups of F-isocrystals. We mainly focus on the proof of Crew's parabolicity conjecture along with its local enhancement for formal neighborhoods of algebraic varieties. Furthermore, we explain how these results are used in the resolution of Chai–Oort Hecke orbit conjecture for Shimura varieties of Hodge type. This conjecture states that the Hecke orbits are dense in the Newton strata.

Cette thèse fournit une synthèse de nos contributions à l'étude de la filtration par les pentes et les groupes de monodromie des F-isocristaux. Nous nous concentrons principalement sur la démonstration de la conjecture de parabolicité de Crew ainsi que sur son amélioration locale pour les voisinages formels des variétés algébriques. De plus, nous expliquons comment ces résultats sont utilisés dans la résolution de la conjecture des orbites de Hecke de Chai–Oort pour les variétés de Shimura de type Hodge. Cette conjecture affirme que les orbites de Hecke sont denses dans les strates de Newton.

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ON THE p-ADIC SLOPE FILTRATION

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Contents

1. Introduction

Dans cette thèse, nous explorons les avancées récentes dans l'étude de la filtration par les pentes des F-isocristaux, en nous concentrant particulièrement sur la conjecture de parabolicité et son amélioration locale. Ces résultats constituent un nouveau lien entre les motifs et la géométrie p-adique. Nous appliquons spécifiquement ces outils pour étudier la géométrie des variétés de Shimura en caractéristique positive.

1.1. Faisceaux lisses et F-isocristaux.

Soient p et ℓ deux nombres premiers distincts. Une grande partie de la géométrie arithmétique en caractéristique p repose sur la *cohomologie étale ℓ-adique*, où les *faisceaux lisses* jouent un rôle central. Cette théorie a été essentielle pour prouver des résultats majeurs, comme l'hypothèse de Riemann sur les corps finis. La variante p-adique des faisceaux lisses est la catégorie des *F-isocristaux*.

Contrairement aux faisceaux lisses ℓ-adiques, les F-isocristaux se présentent sous différentes formes, dont les deux principales catégories sont les F-isocristaux "classiques" de Grothendieck et les *F-isocristaux surconvergents* de Berthelot (voir §2 ou [Ked22] pour un aperçu). Ces derniers se sont avérés être l'analogue "correct" des systèmes locaux ℓ-adiques. Les F-isocristaux surconvergents possèdent des groupes de cohomologie de dimension finie, une *théorie des poids* [Ked06b], et ils satisfont une *correspondance de Langlands* [Abe18]. En combinant la correspondance d'Abe avec la correspondance de Langlands classique pour GL_r sur un corps de fonctions, prouvée par Drinfeld et L. Lafforgue, on obtient, sur les courbes lisses sur un corps fini, une correspondance entre les faisceaux lisses ℓ-adiques semi-simples et les F-isocristaux surconvergents semi-simples. L'assignation préserve les traces du Frobenius aux points fermés. Deux objets ainsi reliés sont appelés *compagnons* (voir [D'Ad20] pour plus de détails).

Les F-isocristaux de Grothendieck présentent des comportements sensiblement différents. Soit X une variété lisse sur un corps parfait k de caractéristique p. Comme démontré par Kedlaya dans [Ked04b], la catégorie des F-isocristaux surconvergents sur X admet un foncteur pleinement fidèle

$$\alpha\colon \text{F-Isoc}^{\dagger}(X) \to \text{F-Isoc}(X).$$

Nous disons que $(\mathcal{M}, \Phi_{\mathcal{M}}) \in \text{F-Isoc}(X)$ est *$\dagger$-prolongeable* s'il est dans l'image essentielle de α et nous écrivons $(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}^{\dagger}})$ pour le F-isocristal surconvergent associé. Pour $(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}^{\dagger}}) \in \text{F-Isoc}^{\dagger}(X)$, l'image dans F-Isoc($X$) acquiert en général plus de sous-quotients. Par exemple, si $(\mathcal{M}, \Phi_{\mathcal{M}})$ a des pentes constantes, il admet une (unique) filtration dans F-Isoc(X)

$$0 = S_0(\mathcal{M}) \subsetneq S_1(\mathcal{M}) \subsetneq \cdots \subsetneq S_m(\mathcal{M}) = \mathcal{M},$$

où, pour chaque $i \geq 0$, le quotient $S_i(\mathcal{M})/S_{i-1}(\mathcal{M})$ est de pente $q_i \in \mathbb{Q}$ et la suite $\{q_i\}_{1\leq i\leq m}$ est croissante. Cette filtration est appelée *filtration par les pentes* (voir §3, [And09]). En général, cette filtration ne se scinde pas et les sous-objets ne sont pas †-prolongeables. Dans les situations géométriques, le premier niveau de la filtration correspond à la cohomologie étale p-adique, qui constitue seulement une partie de la cohomologie cristalline et rigide.

1.2. **La conjecture de parabolicité de Crew.** Les catégories présentées ci-dessus sont tannakiennes, donc équivalentes (éventuellement après extension du corps de base) à la catégorie des représentations linéaires d'un groupe pro-algébrique. L'image de la représentation associée à un objet est ce que l'on appelle le *groupe de monodromie (algébrique)* de l'objet. Dans le cas des systèmes locaux ℓ-adiques, les groupes de monodromie ont déjà été largement étudiés. Pour les catégories des F-isocristaux, beaucoup moins de choses étaient connues. Dans le cas †-prolongeable, une caractéristique intéressante est l'interaction entre le groupe de monodromie comme F-isocristal et F-isocristal surconvergent. Ce phénomène n'a pas d'analogue ℓ-adique.

Soit $(\mathcal{M}, \Phi_{\mathcal{M}})$ un F-isocristal †-prolongeable sur X qui admet la filtration par les pentes. Notons $G(\mathcal{M})$ et $G(\mathcal{M}^\dagger)$ les groupes de monodromie associés. Dans [D'Ad23], nous avons prouvé le résultat suivant (voir §4).

Theorem 1.2.1 (Conjecture de parabolicité de Crew, Théorème 4.1.1)**.** *Le groupe $G(\mathcal{M})$ est le sous-groupe de $G(\mathcal{M}^\dagger)$ qui stabilise la filtration par les pentes. De plus, lorsque $\mathcal{M}^\dagger$ est semi-simple, $G(\mathcal{M})$ est un sous-groupe parabolique de $G(\mathcal{M}^\dagger)$.*

En combinant ce théorème avec le résultat principal de [D'Ad20], on peut calculer le groupe $G(\mathcal{M})$ dans de nombreuses situations géométriques. Comme nous le verrons dans ce texte, les applications les plus intéressantes obtenues jusqu'à présent sont celles où $\mathcal{M}$ est l'isocristal universel de la réduction modulo p d'une variété de Shimura.

Dans [DvH22], nous avons prouvé une sorte d'amélioration du Théorème 1.2.1, que nous présentons en §5. Soit x un point fermé de X et soit $X^{/x}$ la complétion formelle de X en x. Nous écrivons $\mathcal{M}^{/x}$ pour la restriction de $\mathcal{M}$ à $X^{/x}$.

Theorem 1.2.2 (D'A–van Hoften, Théorème 5.4.4)**.** *Si $(\mathcal{M}, \Phi_{\mathcal{M}})$ provient d'un isocristal surconvergent semi-simple muni d'une structure de Frobenius avec polygone de Newton constant, alors*

$$G(\mathcal{M}^{/x}) = R_u(G(\mathcal{M})).$$

Notez que l'énoncé du Théorème 5.4.4 est très éloigné du comportement ℓ-adique. En effet, le groupe fondamental étale géométrique de $X^{/x}$ est trivial, ce qui implique que les faisceaux lisses ℓ-adiques sur $X^{/x}$ ont une monodromie géométrique triviale. Dans la §5.1, nous expliquerons pourquoi le Théorème 1.2.2 doit être considéré comme une amélioration du Théorème 1.2.1. En général, nous nous attendons que la forme plus forte suivante du Théorème 1.2.1 soit vraie.

Conjecture 1.2.3 (D'A–van Hoften)**.** *Si $(\mathcal{M}, \Phi_{\mathcal{M}})$ admet la filtration par les pentes, alors $G(\mathcal{M}^{/x})$ est le noyau de $G(\mathcal{M}) \twoheadrightarrow G(\mathcal{N})$, où $\mathcal{N}$ est l'objet gradué associé à la filtration par les pentes.*

1.3. **Conjecture d'orbite de Hecke.** Comme application de la conjecture de parabolicité et du Théorème 1.2.2, nous avons résolu avec van Hoften une conjecture proposée par Chai et Oort sur le comportement des orbites de Hecke des variétés de Shimura, [DvH22]. Rappelons la conjecture.

Soit p un nombre premier et g un entier positif. Le Problème 15 de la liste de problèmes ouverts de Oort en géométrie algébrique en 1995, [Oor19], était la conjecture suivante.

Conjecture 1.3.1. *Soit $x = (A_x, \lambda)$ un point $\overline{\mathbb{F}}_p$ de l'espace de modules $\mathcal{A}_g$ des variétés abéliennes principalement polarisées de dimension g sur $\overline{\mathbb{F}}_p$. L'orbite de Hecke de x, constituée de tous les points $y \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ paramétrant des variétés abéliennes principalement polarisées reliées à (A_x, λ) par des isogénies symplectiques, est dense pour la topologie de Zariski dans le stratum de Newton de $\mathcal{A}_g$ contenant x.*

Il existe une version raffinée de la Conjecture 1.3.1, également due à Oort, qui considère l'orbite de Hecke premier à p de x, constituée de tous les $y \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ reliés à x par des isogénies symplectiques premier à p. Dans ce cas, le groupe p-divisible quasi-polarisé $(A_x[p^\infty], \lambda)$ est constant sur les orbites de Hecke premier à p (non seulement constant à isogénie près). Par conséquent, l'orbite de Hecke premier à p de x est contenue dans la feuille centrale

$$C(x) = \left\{y \in \mathcal{A}_g(\overline{\mathbb{F}}_p) \mid A_y[p^\infty] \simeq_\lambda A_x[p^\infty]\right\},$$

où $\simeq_\lambda$ désigne un isomorphisme symplectique. Oort a prouvé dans [Oor04] que $C(x)$ est une sous-variété fermée et lisse du stratum de Newton de $\mathcal{A}_g$ contenant x. Il a également conjecturé que l'orbite de Hecke premier à p de x était dense pour la topologie de Zariski dans la feuille centrale $C(x)$. Cette conjecture est connue sous le nom de *conjecture d'orbite de Hecke* pour $\mathcal{A}_g$. Grâce à la formule de produit de Mantovan–Oort, la conjecture d'orbite de Hecke implique la Conjecture 1.3.1.

Les feuilles centrales et les orbites de Hecke premier à p peuvent également être définies pour les fibres spéciales des variétés de Shimura de type Hodge en des premiers de bonne réduction grâce aux travaux de Hamacher et Kim. La conjecture d'orbite de Hecke pour les variétés de Shimura de type Hodge prédit alors que les orbites de Hecke premier à p des points sont denses pour la topologie de Zariski dans les feuilles centrales les contenant (voir [KS23, Ques. 8.2.1] et [Cha06b, Conj. 3.2]). La conjecture d'orbite de Hecke se divise naturellement en une partie discrète et une partie continue. La partie discrète affirme que l'orbite de Hecke premier à p de x intersecte chaque composante connexe de $C(x)$, tandis que la partie continue affirme que l'adhérence de Zariski de l'orbite de Hecke premier à p de x est de la même dimension que $C(x)$. La partie discrète de la conjecture est le Théorème C de [KS23] (voir [vHX21] pour des résultats connexes).

1.3.1. Soit (G, X) un datum de Shimura de type Hodge avec le corps réflexe E, et supposons pour simplifier que G^{ad} est $\mathbb{Q}$-simple. Soit $p > 2$ un nombre premier tel que $G = \mathsf{G} \otimes \mathbb{Q}_p$ est quasi-déployé et déployé sur une extension non ramifiée, soit $U_p \subseteq G(\mathbb{Q}_p)$ un sous-groupe hyperspécial, et soit $U^p \subseteq \mathsf{G}(\mathbb{A}_f^p)$ un sous-groupe ouvert compact suffisamment petit. Choisissons une place v de E divisant p et posons $E = \mathsf{E}_v$. Soit $\mathbf{Sh}_U(\mathsf{G}, \mathsf{X})$ le modèle canonique de variété de Shimura pour (G, X) de niveau $U := U^p U_p$ sur E. Soit $\mathscr{S}_U(\mathsf{G}, \mathsf{X})$ le modèle intégral canonique sur $\mathcal{O}_E$ construit dans [Kis17], et soit $\mathrm{Sh}_{\mathsf{G},U}$ sa fibre spéciale géométrique. Soit $C \subseteq \mathrm{Sh}_{\mathsf{G},U}$ une feuille centrale telle que construite dans [Ham19] (cf. [Kim19]).

Theorem 1.3.2 ([DvH22]). *Si $p \neq 2$ et $Z \subseteq C$ est une sous-variété fermée réduite non vide qui est stable sous les opérateurs de Hecke premier à p, alors $Z = C$.*

Lorsque $\mathrm{Sh}_{G,U}$ est une variété modulaire de Siegel, ce résultat est dû à Chai–Oort et apparaîtra dans leur livre, [CO24]. Leurs preuves ne se généralisent pas aux variétés de Shimura plus générales, car elles reposent sur l'existence de points hypersymétriques dans les strates de Newton, ce qui est généralement faux pour les variétés de Shimura de type Hodge. De plus, leur preuve de la partie continue de la conjecture repose sur le fait que tout point $x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ est contenu dans une grande variété modulaire de Hilbert, et ils utilisent les travaux de Chai–Oort–Yu sur la conjecture d'orbite de Hecke pour les variétés modulaires de Hilbert en des premiers (éventuellement ramifiés). Il existe de nombreux autres résultats partiels, par exemple pour les orbites de Hecke premier à p des points hypersymétriques dans le cas PEL, [Xia20], ou pour les orbites de Hecke premier à p des points μ-ordinaires, [Cha06a], [Sha16], [MST22], [Zho23], [vH24].

Nous avons également prouvé que les classes d'isogénie sont denses dans les strates de Newton qui les contiennent, voir [DvH22, Thm. 8.4.1]. De plus, nous avons obtenu des résultats sur les orbites de Hecke ℓ-adiques pour les premiers $\ell \neq p$ généralisant les travaux de Chai, [Cha11], dans le cas de Siegel, voir [DvH22, Thm. 8.6.1].

Remark 1.3.3. L'hypothèse selon laquelle G^{ad} est $\mathbb{Q}$-simple peut être assouplie au prix d'introduire plus de notations, voir [DvH22, Thm. 8.3.2] pour un énoncé précis. L'hypothèse $p > 2$ est héritée des travaux de Kim [Kim19], et n'est pas nécessaire pour les variétés modulaires de Siegel.

Remark 1.3.4. Bragg–Yang ont prouvé un critère de bonne réduction potentielle pour les surfaces K3, voir [BY23, Thm. 8.10], sous réserve de la conjecture d'orbite de Hecke pour certaines variétés de Shimura orthogonales, voir [*ibid.*, Conj. 8.2]. Dans [DvH22, §8.5], nous avons expliqué que nos résultats peuvent être utilisés pour prouver cette conjecture pour $p > 2$.

1.3.2. Pour résoudre la conjecture d'orbite de Hecke, nous avons principalement exploité une stratégie proposée par Chai et Oort. Grâce à [KS23], pour prouver la conjecture, il suffisait de montrer que la dimension de Z est maximale. Ainsi, nous avons pu examiner le voisinage formel $Z^{/x}$ par rapport à un certain point fermé lisse $x \in Z$. Sur ce schéma formel, nous avons combiné [vH24, Cor. 3.3.3] et le Théorème 1.2.2 pour prouver que le groupe de monodromie de l'isocristal universel F est maximal[1]. Pour en déduire qu'une grande monodromie garantit un grand voisinage formel, nous avons combiné un résultat de rigidité de Chai–Oort sur les sous-schémas formels *fortement Tate-linéaires*, [CO22], et une borne de monodromie obtenue en utilisant les champs de Cartier–Witt (Théorème 6.5.1). Le Théorème 6.5.1 est la "moitié" d'une conjecture que nous avons formulée sur la détermination des groupes de monodromie pertinents (voir Conjecture 6.4.2).

1.4. **Structure de la thèse.** Dans la §2, nous présentons de manière historique les principales théories de cohomologie p-adique en caractéristique positive. Dans la §3, nous rappelons d'abord la notion de filtration par les pentes des F-isocristaux, puis nous examinons le comportement des F-isocristaux sur les *schémas Frobenius-lisses* et expliquons la preuve d'un nouveau résultat de pleine fidélité pour le foncteur de restriction au point générique (Théorème 3.3.4). Nous l'utilisons

[1]Notez que [vH24, Cor. 3.3.3] utilise indirectement la correspondance de Langlands en caractéristique positive car il repose sur [D'Ad20].

pour prouver un résultat général sur l'existence de la filtration par les pentes (Théorème 3.3.6). Ensuite, nous rappelons les définitions et théorèmes principaux de la théorie des groupes p-divisibles. Nous terminons la section en introduisant les *anneaux quasi-réguliers sémiparfaits* et la notion de groupes p-divisibles *complètement divisibles par les pentes*. Dans la §4, nous présentons la preuve de la conjecture de parabolicité, dans la §5 son amélioration locale, et dans la §6 nous détaillons comment ces résultats contribuent à résoudre la conjecture d'orbite de Hecke. Dans la §6.2, nous rappelons la définition des *algèbres de Dieudonné–Lie* que nous avons utilisées pour définir certaines *coordonnées de Serre–Tate généralisées*. Dans la §7, nous passons en revue d'autres applications de la conjecture de parabolicité. En particulier, dans la §7.3, nous présentons quelques variantes p-adiques de la conjecture de Tate et sa relation avec la conjecture de parabolicité.

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2. Review of p-adic cohomology theories

In this section we shall present an historical overview of various p-adic cohomology theories that have been developed over time.

2.1. **Algebraic de Rham cohomology.** Let us first recall the situation over the complex numbers. Let X be a smooth algebraic variety over $\mathbb{C}$. One can construct on X a complex of sheaves of algebraic differential forms

$$\Omega^\bullet_{X/\mathbb{C}} \coloneqq 0 \to \mathcal{O}_{X/\mathbb{C}} \xrightarrow{d} \Omega^1_{X/\mathbb{C}} \xrightarrow{d} \Omega^2_{X/\mathbb{C}} \xrightarrow{d} \dots .$$

The algebraic de Rham cohomology $H^\bullet_{\mathrm{dR}}(X/\mathbb{C})$ is defined to be the hypercohomology of the complex $\Omega^\bullet_{X/\mathbb{C}}$ as a complex of sheaves for the Zariski topology of X. If X is affine these groups coincide with the cohomology groups of the complex of global sections of $\Omega^\bullet_{X/\mathbb{C}}$. Grothendieck proved that for smooth varieties, de Rham cohomology computes the singular cohomology of the complex manifold X^{an} associated to X. For example, for the affine line $\mathbb{A}^1_{\mathbb{C}}$ (corresponding to the complex manifold $\mathbb{C}$), we have that $H^1_{\mathrm{dR}}(X/\mathbb{C}) = 0$ since

$$\mathbb{C}[x] \simeq H^0(X, \mathcal{O}_{X/\mathbb{C}}) \xrightarrow{d} H^0(X, \Omega^1_{X/\mathbb{C}}) \simeq \mathbb{C}[x]dx$$

is surjective.

The construction of the groups $H^\bullet_{\mathrm{dR}}(X/\mathbb{C})$ is completely algebraic, thus one can define them for every smooth algebraic variety over a field k (or more generally for smooth schemes over a base S). In characteristic 0, this construction works well and the cohomology groups have all the desired properties. In particular, the cohomology groups are finite-dimensional k-vector spaces. In positive characteristic one encounters some problems. If $\mathbb{A}^1_{\mathbb{F}_p}$ is the affine line over $\mathbb{F}_p$, the $\mathbb{F}_p$-vector space $H^1_{\mathrm{dR}}(\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p)$ is infinite-dimensional. The differential forms $x^{p-1}dx, x^{2p-1}dx, \dots$ define an infinite sequence of linearly independent classes. Even for smooth and proper varieties there are some undesired phenomena. For example, the dimension of $H^1_{\mathrm{dR}}(X/k)$ might be bigger than twice the dimension of the Picard variety of X.

2.2. **Dwork and Monsky–Washnitzer cohomology.** A way to overcome the problems of algebraic de Rham cohomology in positive characteristic is to work with lifting of the varieties to characteristic 0. If k is a perfect field of positive characteristic p, a very convenient way to pass to characteristic 0 is by considering the ring of Witt-vectors W. This ring is a p-adically complete discrete valuation ring with residue field k and fraction field K of characteristic 0. The construction of this ring is functorial in k and it generalises the construction of the ring of p-adic integers $\mathbb{Z}_p$, which is the ring of Witt-vectors of $\mathbb{F}_p$. Even if there are many different ways to lift the equations of an algebraic variety over k to W, if X is the reduction modulo p of two different smooth and proper schemes X_∞ and X'_∞ over W, the de Rham cohomology groups $H^i_{\mathrm{dR}}(X_\infty/W)$ and $H^i_{\mathrm{dR}}(X'_\infty/W)$ are canonically isomorphic. In this case, we also have that the W-module $H^i_{\mathrm{dR}}(X_\infty/W)$ is finitely generated.

Dwork was the first one to construct and use p-adic cohomology groups endowed with an *action of the Frobenius* for some particular algebraic varieties over k. Many authors, influenced by Dwork's ideas, tried to construct an entire cohomology theory for algebraic varieties over perfect fields of positive characteristic with coefficients in W. One of the first difficulties is that, in general, smooth and proper varieties over k do not lift to smooth proper schemes over W. If X is smooth and affine, instead, there is always a formal lift $\mathfrak{X} \coloneqq \mathrm{Spf}(A_\infty)$ over $\mathrm{Spf}(W)$, where A_∞ is a p-adically complete topologically finitely generated and formally smooth W-algebra. Even for this lift one has a natural de Rham complex

$$\Omega^\bullet_{\mathfrak{X}/W} \coloneqq \mathcal{O}_{\mathfrak{X}/W} \xrightarrow{d} \Omega^1_{\mathfrak{X}/W} \xrightarrow{d} \Omega^2_{\mathfrak{X}/W} \xrightarrow{d} \dots .$$

The A_∞-module of global sections $\Gamma(\Omega^i_{\mathfrak{X}/W})$ is defined to be the projective limit $\varprojlim \Omega^i_{A_n/W_n}$ where $A_n \coloneqq A_\infty/p^n$ and $W_n \coloneqq W/p^n$. One can prove that the cohomology of $\Omega^\bullet_{\mathfrak{X}/W}$ depends only on X, again, but in this case the cohomology groups are huge. This is already clear for the affine line. If $\mathfrak{X} \coloneqq \hat{\mathbb{A}}^1_W$ is the formal affine line over W, its ring of global sections $\varprojlim_n W_n[x]$ is isomorphic to the ring

$$W\langle x\rangle \coloneqq \left\{ \sum_{i=0}^{\infty} a_i x^i \in W[[x]] \;\middle|\; v_p(a_i) \to \infty \text{ when } i \to \infty \right\}.$$

Therefore, we get the complex

$$W\langle x\rangle \xrightarrow{d} W\langle x\rangle dx.$$

In this case, there are lots of undesirable classes. The first type is given by those classes represented by the differential forms like $x^{ip-1}dx$, which cannot be integrated because p is not invertible. These differential forms can be integrated after multiplying enough times by p, thus they define torsion classes of H^1. The second type of issue is given by the differential forms like $\sum_{i=0}^\infty p^i x^{p^i-1}dx$. This differential form cannot be integrated because the formal integral is given by the series $\sum_{i=0}^\infty x^{p^i}+c$, which is not in $W\langle x\rangle$. In this case, one cannot integrate the differential form even after multiplying by a big power of p. Note also that the class represented by $\sum_{i=0}^\infty p^i x^{p^i-1}dx$ is infinitely p-divisible, since for every $n \geq 1$ it can be written as

$$\left(\sum_{i=0}^{n-1} p^i x^{p^i-1}dx\right) + \left(p^n \sum_{i=n}^{\infty} p^{i-n} x^{p^i-1}dx\right)$$

and the first summand is exact, while the second is divisible by p^n.

This problem was partially solved by Dwork by replacing $W\langle x\rangle$ with the *ring of overconvergent series*

$$W\langle x\rangle^\dagger \coloneqq \left\{ \sum_{i=0}^{\infty} a_i x^i \in W[[x]] \;\middle|\; \left(v_p(a_i) - \tfrac{i}{n}\right) \to \infty \text{ for some } n > 0 \text{ and } i \to \infty \right\}.$$

Geometrically, the ring $W\langle x\rangle^\dagger$ is a ring of series such that the p-adic radius of convergence is strictly bigger than 1, whereas $W\langle x\rangle$ is a ring of series with p-adic radius equal to 1. After passing to $K\langle x\rangle^\dagger \coloneqq W\langle x\rangle^\dagger[\frac{1}{p}]$, one can easily check that $K\langle x\rangle^\dagger \xrightarrow{d} K\langle x\rangle^\dagger dx$ is surjective, so that the complex

$$K\langle x\rangle^\dagger \xrightarrow{d} K\langle x\rangle^\dagger dx$$

is quasi-isomorphic to $K[0]$, as it happens over $\mathbb{C}$.

The idea of working with overconvergent series has been used by Monsky–Washnitzer to define a cohomology theory for smooth affine varieties over k. For a smooth algebra A over k, they construct a certain flat W-algebra $A^\dagger_\infty$, playing the role of $W\langle x\rangle^\dagger$, such that $A^\dagger_\infty/p \simeq A$. They define $H^\bullet_{\mathrm{MW}}(\mathrm{Spec}(A)/W)$ as the cohomology groups of

$$A^\dagger \xrightarrow{d} \widetilde{\Omega}^1_{A^\dagger_\infty/W} \xrightarrow{d} \widetilde{\Omega}^2_{A^\dagger_\infty/W} \to \dots$$

with $\widetilde{\Omega}^i_{A^\dagger_\infty/K}$ the $A^\dagger_\infty$-submodules of $\Omega^i_{A^\dagger_\infty/K}$ of p-adically continuous differential forms. They prove that this construction is functorial in A and does not depend on the lift. The groups $H^\bullet_{\mathrm{MW}}(\mathrm{Spec}(A)/W)[\frac{1}{p}]$ are finite-dimensional vector space, as proved by Mebkhout in [Meb97].

2.3. **Crystalline cohomology.** Grothendieck in 1966 introduced *crystalline cohomology.* His point of view was quite different from the one of Dwork and Monsky–Washnitzer. Rather than working explicitly with differential forms, he has constructed the crystalline cohomology groups using a Grothendieck site, the *crystalline site* [Gro68]. The idea behind this construction is that, even if the varieties over k do not have good lifts to characteristic 0, they do lift nicely locally. We can say that, roughly speaking, the crystalline site contains all these lifts. The name comes from a fitting similitude Grothendieck found with common crystals.

> *"A crystal has two characteristic properties: the rigidity, and the faculty to grow in an adequate neighbourhood. There are crystals of all kind of substances: crystals of soda, of sulphur, of modules, of rings, of relative schemes etc."*

Let us briefly recall Grothendieck's construction. If X is a variety over k, he considered the category $\mathrm{Cris}(X)$ of triples (U,T,γ) where $U\subseteq X$ is a Zariski open, $U\hookrightarrow T$ is a nilpotent thickening over W defined by an ideal sheaf $\mathcal{I}$, and γ is a divided power structure [2] on $\mathcal{I}$ such that $\gamma_n(p)=p^n/n!$ for every $n\geq 1$. The category $\mathrm{Cris}(X)$, endowed with its natural Zariski topology, is called the *crystalline site* of X (with respect to W). The category X_{cris} of sheaves of sets of $\mathrm{Cris}(X)$ is a ringed topos with ring object $\mathcal{O}_{X_{\mathrm{cris}}}$, the sheaf that assigns at each triple (U,T,γ) the set of global sections $\Gamma(T,\mathcal{O}_T)$. The crystalline cohomology groups of X are then defined as $H^\bullet_{\mathrm{cris}}(X/W) := H^\bullet(X_{\mathrm{cris}},\mathcal{O}_{X_{\mathrm{cris}}})$.

Crystalline cohomology has been developed chiefly by Berthelot and Ogus (see [Ber74] and [BO78]) who proved that the crystalline cohomology groups of smooth and proper varieties are finitely generated and satisfy Poincaré duality, the Künneth formula, and the Lefschetz trace formula for Frobenius. Moreover, there is a theory of crystalline Chern classes and cycle classes, developed by Berthelot–Illusie [BI70] and Gillet–Messing [GM87].

Given a sheaf $\mathcal{F}$ on $\mathrm{Cris}(X)$ and an object $(U,T,\gamma)\in\mathrm{Cris}(X)$, there is a natural way to construct a Zariski sheaf $\mathcal{F}_T$ over T. This is defined as $\mathcal{F}_T(W) := \mathcal{F}(U\cap W, W,\gamma)$ for every Zariski open $W\subseteq T$.

Definition 2.3.1. A *crystal* over X is a sheaf over $\mathrm{Cris}(X)$ satisfying the following two conditions.

(1) For every $(U,T,\gamma)\in\mathrm{Cris}(X)$, the sheaf $\mathcal{F}_T$ is quasi-coherent.

[2] A divided structure on an ideal I is the datum of map of sets $\gamma_n : I\to I$ for $n\geq 1$ satisfying all the algebraic relations of the maps $\gamma_n:(x)\to(x)$ of the ideal $(x)\subseteq\mathbb{Q}[x]$ defined by $\gamma_n(f)=f^n/n!$ for $n\geq 1$. It is natural to ask that $\mathcal{I}$ has a divided power structure in order to construct Chern classes of line bundles in crystalline cohomology.

(2) For every morphism $f\colon (U,T,\gamma)\to(U',T',\gamma')$ in $\mathrm{Cris}(X)$, the comparison morphism
$$f^*\mathcal{F}_{T'}\to\mathcal{F}_T$$
is an isomorphism.

Crystals are the coefficients of crystalline cohomology. It is often useful to consider also their rational variant, called *isocrystals*. As proven by Morrow [Mor19], isocrystals parametrise the variation of rational crystalline cohomology groups of smooth and proper families of varieties. In this situation, they carry a natural *Frobenius structure*, given by the variation of the action of the Frobenius on the different crystalline cohomology groups. For this reason, for geometric applications, it is often convenient to work with the category of *F-crystals* (resp. *F-isocrystals*), which are crystals (resp. isocrystals) endowed with a Frobenius structure.

Even if the theory works well for smooth and proper varieties, the problems explained in §2.2 for open smooth varieties remain unchanged. Indeed, if X lifts to a smooth formal scheme $\mathfrak{X}$ over W, then $H^\bullet_{\mathrm{cris}}(X/W)=H^\bullet_{\mathrm{dR}}(\mathfrak{X}/W)$. In particular, the cohomology of $\mathbb{A}^1_k$ is not of finite rank.

2.4. **Rigid cohomology.** In 1996 Berthelot defined another variant of crystalline cohomology, called *rigid cohomology* [Ber96b]. The idea is to embed X Zariski-locally in some smooth formal scheme $\mathfrak{P}$ and then consider the de Rham cohomology of a certain decreasing chain of p-adic tubular neighbourhoods of X in the rigid generic fibre $\mathfrak{P}_K$. These tubes are chosen in order to avoid those series which are not overconvergent. The geometric idea behind this construction is that if you have a manifold $\mathcal{M}$ embedded in $\mathbb{R}^n$, every small tubular neighbourhood of $\mathcal{M}$ has the same singular cohomology as $\mathcal{M}$.

The rigid cohomology groups $H^\bullet_{\mathrm{rig}}(X/K)$ are vector spaces over K. Berthelot proved that if X is smooth and proper, then $H^\bullet_{\mathrm{rig}}(X/K)=H^\bullet_{\mathrm{cris}}(X/W)[\frac{1}{p}]$ and if X is smooth and affine $H^\bullet_{\mathrm{rig}}(X/K)=H^\bullet_{\mathrm{MW}}(X/W)[\frac{1}{p}]$. Thanks to this, he deduced that the rigid cohomology groups (with constant coefficients) of smooth varieties are finite-dimensional [Ber97]. This result has been extended to general varieties in [GK02] and [Tsu03]. Rigid cohomology has many desirable properties as Poincaré duality and the Künneth formula and it also admits a theory of Chern classes and cycle classes, developed by Petrequin [Pet03].

The natural category of coefficients of rigid cohomology is the category of *overconvergent isocrystals*. The finiteness of rigid cohomology with coefficients in an overconvergent isocrystal was proven by Kedlaya in [Ked06a]. In this case, the proof relies on *Crew's local monodromy conjecture*, proved independently by André [And02], Mebkhout [Meb02], and Kedlaya [Ked04a]. Using Crew's conjecture, Kedlaya proved also a comparison theorem for convergent and overconvergent F-isocrystals, called *Kedlaya's full faithfulness theorem* [Ked04b]. Berthelot's conjecture, remains the main open problem in the theory (see [Laz16], [EV24]). It says that the higher direct image of an overconvergent F-isocrystal via a smooth and proper morphism of smooth varieties is again an overconvergent F-isocrystal.

A variant of the theory, also introduced by Berthelot, uses the concept of *arithmetic D-modules*, and it has been developed in [Ber96a] and [Ber00]. This theory has better functoriality properties. Thanks to *Kedlaya's semistable reduction theorem*, [Ked11], there has been a great development (see [CT12] and [AC18]). This culminated to the proof of the Langlands correspondence for overconvergent F-isocrystals by Abe, [Abe18].

Finally, it is worth mentioning that in 2006 Le Stum introduced a site-theoretic approach to defining rigid cohomology, [LeS11]. Its definition remains in the realm of p-adic geometry.

2.5. **Other theories.** Ogus in [Ogu84] introduced a variant of the crystalline site, called the *convergent site*. This was motivated by Dwork's observation that the isocrystals that can be endowed with a Frobenius structure have an additional "local convergence property". The convergent site produces a smaller category of isocrystals, called *convergent isocrystals*, with better proprieties. Nonetheless, even in this case, the cohomology groups one obtains are not finitely generated for open smooth varieties.

The theory of *log crystalline cohomology* has been introduced by Kato in [Kat89]. This is an integral variant of crystalline cohomology which has some additional nice properties. For example, it is finitely generated for smooth varieties admitting a smooth compactification with a normal crossing divisor at infinity. Log crystalline cohomology has a category of coefficients, called *log-(iso)crystals*.

Kramer-Miller constructed in 2016 the category of *F-isocrystals with log-decay* (see [KM16]) motivated by some ideas of Dwork. At the moment, he built a complete theory over smooth curves, while in higher dimension he gave only some partial definitions "in coordinates". With this he managed to give a new proof of a theorem of Drinfeld–Kedlaya on the slopes of overconvergent F-isocrystals [KM19]. Wan conjectured that there should be a "log-decay" crystalline cohomology theory associated to these F-isocrystals.

2.6. **Edged crystalline cohomology.** Recently we constructed in [D'Ad24b] a new cohomology theory, which interpolates crystalline, log crystalline, and rigid cohomology and recover Kramer-Miller's F-isocrystals with log-decay. This theory depends on the choice of a superadditive map of sets $\tau : \mathbb{N}_{>0} \to \mathbb{N} \cup \{\infty\}$, which is called *edge type*. For each edge type τ, the associated cohomology theory is called *τ-edged crystalline cohomology*.

More precisely, for every *marked scheme*[3] $\underline{X}$, we associated the *marked crystalline site* $\mathrm{Cris}(\underline{X})$, of triples $(\underline{U}, T, \gamma)$ as above, with the difference that $\underline{U} \subseteq \underline{X}$ is an open marked subscheme. For each edge type τ we then constructed a sheaf of algebras $\mathcal{O}^{\tau}_{\underline{X}_{\mathrm{cris}}}$ over $\mathrm{Cris}(\underline{X})$. The sheaf $\mathcal{O}^{\tau}_{\underline{X}_{\mathrm{cris}}}$ is a sort of localisation of the sheaf $\mathcal{O}_{X_{\mathrm{cris}}}$, called *$\tau$-edged localisation*, where we add poles whose order is "p-adically small". This construction depends on τ. The geometric picture behind the name is that rather than making a neat cut to the scheme by removing the marking, we leave an edge of type τ.

For an algebraic variety X over a field k, we defined

$$H^{\bullet}_{\tau\text{-cris}}(X/K) := \varinjlim_{X \subseteq \underline{Y}} H^{\bullet}(\mathrm{Cris}(\underline{Y}), \mathcal{O}_{\underline{Y}_{\mathrm{cris}}})[\tfrac{1}{p}],$$

where $\underline{Y}$ varies among the proper marked varieties containing X outside the marking. These vector spaces are endowed with natural integral lattices which depend on the choice of the compactification.

[3] A marked scheme is roughly speaking a scheme with the choice of an effective Cartier divisor at infinity.

The edge type $\tau(e) = e$ corresponds to rigid cohomology, exponential edge types as $\tau(e) = p^e$ correspond to Kramer-Miller's theory, and the edge type $\tau(e) \equiv \infty$ correspond to crystalline cohomology. Note that the structural sheaves are constructed in such a way that $\mathcal{O}^{\tau_1}_{\underline{X}_{\mathrm{cris}}} \subseteq \mathcal{O}^{\tau_2}_{\underline{X}_{\mathrm{cris}}}$ when $\tau_1 \leq \tau_2$.

Besides recovering previous theories, with edged crystalline cohomology it is possible to get the conjectured cohomology theory attached to log-decay isocrystals. Another crucial point is the construction of a new integral theory for rigid cohomology with new finitenesses properties that facilitates various geometric operations on the coefficients.

3. Slope filtration and Dieudonné modules

3.1. **Slope filtration over a field.** If k is a perfect field, the category of (coherent) crystals over $\mathrm{Spec}(k)$ is equivalent to the category of finitely-generated $W(k)$-modules. To prove this, it is convenient filtering the category $\mathrm{Cris}(\mathrm{Spec}(k))$ by truncated subcategories. For every $e \geq 0$, we write Σ_e for the PD-scheme $(\mathrm{Spec}(\mathbb{Z}/p^e), (p), \gamma)$ with $\gamma_n(p) = p^n/n!$ for every $n \geq 1$ and we get full subcategories

$$\mathrm{Cris}(\mathrm{Spec}(k)/\Sigma_e) \subseteq \mathrm{Cris}(\mathrm{Spec}(k))$$

consisting of those PD-schemes killed by p^e. Thanks to following lemma, $\mathrm{Spec}(W_e(k))$ endowed with the natural PD-structure on (p) is the final object of $\mathrm{Cris}(\mathrm{Spec}(k)/\Sigma_e)$.

Lemma 3.1.1. *For every surjection $A \twoheadrightarrow k$ with nilpotent kernel, there exists a unique dotted arrow which makes the following solid diagram commute*

$$\begin{tikzcd} & A \arrow[d, two heads] \\ W(k) \arrow[ur, dotted] \arrow[r, two heads] & k. \end{tikzcd}$$

Proof. If $\sigma\colon k \to A$ is any set-theoretic section, one can prove that for every $a \in k$, the sequence $\sigma(a^{1/p^n})^{p^n}$ is eventually constant and independent of σ. This defines a multiplicative section $\tau\colon k \to A$, called *Teichmüller lift*. The section τ can be extended to a ring morphism $W(k) \to A$. □

Definition 3.1.2. For a scheme X of characteristic p, we write $\mathrm{Isoc}(X)$ for the isogeny category of the category of coherent[4] crystals over X. This is called the category of *(coherent) isocrystals* over X. We write $\mathrm{F\text{-}Isoc}(X)$ for the category of isocrystals $\mathcal{M}$ endowed with a Frobenius structure $\Phi_{\mathcal{M}}\colon \mathcal{M}^{(p)} \to \mathcal{M}$. These are the *$F$-isocrystals.*

Thanks to Lemma 3.1.1, the category of isocrystals over $\mathrm{Spec}(k)$ is the category of finite-dimensional $K(k)$-vector spaces, where $K(k) \coloneqq W(k)[\frac{1}{p}]$. Thus, in this case, isocrystals form a very explicit category. When we look at F-isocrystals over a perfect field there is a fundamental special behaviour: the existence of the *slope decomposition.*

To explain the slope decomposition, let us first construct a special class of F-isocrystals, the *elementary* ones.

[4]Note that for us isocrystals will always be coherent.

Construction 3.1.3. Let s, r be coprime integers with $r > 0$. We write $E_{s/r}$ for an r-dimensional vector space with basis $e_1, \cdots, e_r$. Let $\varphi_{E_{s/r}}$ be the semi-linear bijection of $E_{s/r}$ which sends e_i to e_{i+1} for $i \leq r$ and e_r to $p^s e_r$.

Theorem 3.1.4 (Dieudonné, Manin). *If k is algebraically closed, for every F-isocrystal (M, φ_M), there exist positive integers $a_1, \cdots, a_m$ and rational numbers $q_1, \cdots, q_m$ such that*

$$(M, \varphi_M) = \bigoplus_{i=1}^{m} E_{q_i}^{\oplus a_i}.$$

In particular, the category of F-isocrystals over k is semi-simple.

Definition 3.1.5. The numbers q_i are the *slopes* of (M, φ_M) and the dimension of $E_{q_i}^{\oplus a_i}$ is the multiplicity of each slope q_i. The definition of slopes and their multiplicity extends to F-isocrystals over any field (not necessarily perfect), by taking the base change to any algebraically closed field. We say that an F-isocrystal over a field is *isoclinic* if it has only 1 slope (of any multiplicity).

By étale descent, the previous theorem admits the following corollary.

Corollary 3.1.6 (Dieudonné, Manin). *If k is a perfect field, for every F-isocrystal (M, φ_M) over* $\mathrm{Spec}(k)$*, there exist isoclinic F-isocrystals (M_i, φ_{M_i}) such that*

$$(M, \varphi_M) = \bigoplus_{i=1}^{m} (M_i, \varphi_{M_i}).$$

3.1.1. The situation over more general fields is more complicated. The category of isocrystals is not equivalent to a category of vector spaces. The main problem is that we can not construct Teichmüller lifts as in the perfect case and the ring $W(k)$ is not as well behaved as before (for example, $W(k)/p \neq k$). One can construct instead *Cohen rings* Λ such that $\Lambda/p = k$. Nonetheless, there is some indeterminacy given by the fact that two different choices Λ and Λ' are not canonically isomorphic. In other words, there is a non-trivial group of ring automorphisms of Λ which lift the identity of k. The result, is that the category of crystals over k is the category of coherent Λ-modules endowed with a continuous *topologically p-nilpotent connection.*

The slope decomposition of F-isocrystals over the algebraic closure of k does not descend in general to a decomposition over k. Nonetheless, every F-isocrystal over $\mathrm{Spec}(k)$ admits the *slope filtration.*

Theorem 3.1.7 ([Kat79]). *If k is a field, for every F-isocrystal (M, φ_M) over* $\mathrm{Spec}(k)$*, there exists a unique Frobenius-stable filtration $0 = S_0(M) \subsetneq S_1(M) \subsetneq \cdots \subsetneq S_m(M) = M$ such that every quotient $S_{i+1}(M)/S_i(M)$ is isoclinic of some slope q_i and $q_1 < q_2 < \cdots < q_m$.*

The filtration of Theorem 3.1.7 is called the slope filtration of (M, φ_M) and in general it does not split.

Example 3.1.8. The classical example of a non-splitting slope filtration can be constructed when $k = \overline{\mathbb{F}}_p(t)$. The F-isocrystal (M, φ_M) is given by the Dieudonné module of the p-divisible group of an elliptic curve over k with transcendental j-invariant (see §3.4). In this case, the slope filtration induces an exact sequence

$$0 \to N \to M \to Q \to 0$$

with N of rank 1 and slope 0 and Q of rank 1 and slope 1. This sequence does not split.

3.2. **Frobenius-smooth schemes.** In the study of crystals, there is a special class of schemes with a particularly good behaviour.

Definition 3.2.1. We say that a scheme X over $\mathbb{F}_p$ is *Frobenius-smooth* if the absolute Frobenius $F: X \to X$ is syntomic.

The condition of being Frobenius-smooth is equivalent to X being Zariski locally of the form $\mathrm{Spec}(B)$ where B has a *finite (absolute) p-basis*, [Dri22, Lem. 2.1.1], which means that there exist elements $x_1, \cdots, x_n$ such that every element $b \in B$ can be uniquely written as

$$b = \sum b_\alpha^p x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where $0 \leq \alpha_i \leq p-1$ and $b_\alpha \in B$. Thanks to §1.1 of [BM90], a ring B with a p-basis admits a p-adic lift $\tilde{B} \twoheadrightarrow B$. By [*ibid.*, Prop. 1.3.3], the datum of a crystal over $\mathrm{Spec}(B)$ is then equivalent to the datum of a coherent $\tilde{B}$-module M^+ and topologically p-nilpotent derivations of M^+ associated to some choice of a lift of a p-basis of B to $\tilde{B}$.

The main examples of Frobenius-smooth schemes that we will encounter are smooth schemes over perfect fields and power series rings over perfect fields. If X is a Noetherian Frobenius-smooth scheme, then it is regular by a result of Kunz (see Tag 0EC0 of [Stacks]).

Definition 3.2.2. When X is irreducible, Noetherian, and Frobenius-smooth we define

$$\kappa := \bigcap_{i=1}^{\infty} \Gamma(X, \mathcal{O}_X)^{p^n}$$

to be the *field of constants* of X. Note that κ is a field thanks to [Dri22, §3.1.2] and when X is in addition a geometrically connected scheme of finite type over a perfect field, κ coincides with the base field. We also write K for $W(\kappa)[\frac{1}{p}]$.

Proposition 3.2.3 ([Dri22, Cor. 3.3.3])**.** *If X is an irreducible Noetherian Frobenius-smooth scheme, then* $\mathrm{Isoc}(X)$ *is a K-linear Tannakian category.*

This allows us to define the monodromy groups of isocrystals in this situation.

Definition 3.2.4. Let X be an irreducible Noetherian Frobenius-smooth scheme and let $\mathcal{M}$ be an isocrystal over X. We define $\langle \mathcal{M} \rangle$ to be the Tannakian subcategory of $\mathrm{Isoc}(X)$ generated by $\mathcal{M}$. If ξ is an Ω-point of X for some perfect field Ω, we define $G(\mathcal{M}, \xi)$ to be the Tannaka group of $\langle \mathcal{M} \rangle$ with respect to the fibre functor induced by ξ. We call it the *monodromy group of* $\mathcal{M}$ with respect to ξ.

3.3. **Slope filtration over Noetherian Frobenius-smooth schemes.** Let X be an irreducible Noetherian Frobenius-smooth scheme over $\mathbb{F}_p$ with generic point η and let $\mathcal{M}$ be an isocrystal over X such that $F^*\mathcal{M} \simeq \mathcal{M}$. In this section we want to prove that F-isocrystals over X with constant Newton polygon admit the slope filtration. For this purpose, we use a result that allows us to exploit the slope filtration of the restriction to the generic point. We recall the following theorem by de Jong.

Theorem 3.3.1 ([deJ98, Thm. 1.1])**.** *If $X = \mathrm{Spec}(A)$ is affine with A a DVR and $(\mathcal{M}^+, \Phi_{\mathcal{M}^+})$ is a free F-crystal over X of finite rank, then for every $(m,n) \in \mathbb{Z} \times \mathbb{Z}_{>0}$, we have*

$$H^0(\eta, \mathcal{M}^+_\eta)^{F^n=p^m} = H^0(X, \mathcal{M}^+)^{F^n=p^m}.$$

Corollary 3.3.2. *If X is as in Theorem 3.3.1 and $\mathcal{M}$ is an isocrystal over X such that $F^*\mathcal{M} \simeq \mathcal{M}$, then $H^0(\eta, \mathcal{M}_\eta) = H^0(X, \mathcal{M})$.*

Proof. Let $(\mathcal{M}^+, \Phi_{\mathcal{M}^+})$ be an F-crystal such that $\mathcal{M}^+[\frac{1}{p}] = \mathcal{M}$. Since X is of dimension 1, by [Cre92a, Lem. 2.5.1], after possibly replacing $\mathcal{M}^+$ with its double dual we can assume $\mathcal{M}^+$ free. We may further assume that the field of constants of A is algebraically closed thanks to [deJ98, §3]. By Dieudonné–Manin classification both $H^0(\eta, \mathcal{M}_\eta)$ and $H^0(X, \mathcal{M})$ are generated by vectors v such that $\Phi^n_{\mathcal{M}}(v) = p^m v$ for some $m, n \in \mathbb{Z} \times \mathbb{Z}_{>0}$. The result then follows from Theorem 3.3.1. □

De Jong's theorem can be extended to more general irreducible Noetherian Frobenius-smooth schemes by using a Hartogs' argument. We need the following lemma.

Lemma 3.3.3. *Let $f\colon A \to A'$ be an injective morphism of Noetherian Frobenius-smooth rings which sends p-bases to p-bases and write $\tilde{f}\colon \tilde{A} \to \tilde{A}'$ for a p-adic lift of f. For an isocrystal $\mathcal{M}$ over* $\mathrm{Spec}(A)$ *write $\mathcal{M}'$ for the pullback to* $\mathrm{Spec}(A')$ *and M, M' for the associated modules over $\tilde{A}[\frac{1}{p}]$ and $\tilde{A}'[\frac{1}{p}]$. The following diagram is cartesian*

$$\begin{array}{ccc} H^0(\mathrm{Spec}(A), \mathcal{M}) & \longrightarrow & M \\ \downarrow & & \downarrow \\ H^0(\mathrm{Spec}(A'), \mathcal{M}') & \longrightarrow & M'. \end{array} \tag{3.1}$$

Proof. By [Dri22, Prop. 3.5.2], the module M is projective, which implies that $M \to M'$ is injective. We choose a lift $\{\tilde{x}_1, \ldots, \tilde{x}_n\} \subseteq \tilde{A}$ of a p-basis of A. By the assumption, this is sent by $\tilde{f}$ to a lift of a p-basis of A'. This choice then defines differential operators $\partial_1, \ldots, \partial_n$ of M' that stabilise $M \subseteq M'$. By [BM90, Prop. 1.3.3], this implies that $H^0(\mathrm{Spec}(A'), \mathcal{M}') \subseteq M'$ (resp. $H^0(\mathrm{Spec}(A), \mathcal{M}) \subseteq M$) is the subspace of elements killed by $\partial_1, \ldots, \partial_n$. This ends the proof. □

Theorem 3.3.4 ([DvH22, Thm. 3.2.4])**.** *If X is an irreducible Noetherian Frobenius-smooth scheme over $\mathbb{F}_p$ and $\mathcal{N}$ is a subquotient of an isocrystal $\mathcal{M}$ over X such that $F^*\mathcal{M} \simeq \mathcal{M}$, then $H^0(\eta, \mathcal{N}_\eta) = H^0(X, \mathcal{N})$. In particular, the functor* $\text{F-Isoc}(X) \to \text{F-Isoc}(\eta)$ *is fully faithful.*

Proof. By Theorem 5.10 of [DE22], we know that $\mathcal{N}$ is a subobject of some isocrystal $\mathcal{M}'$ such that $F^*\mathcal{M}' \simeq \mathcal{M}'$. Thanks to [*ibid.*, Lemma 5.6], it is then enough to prove the result for an isocrystal $\mathcal{M}$ such that $F^*\mathcal{M} \simeq \mathcal{M}$. By Zariski descent, we may further assume $X = \mathrm{Spec}(A)$ affine.
Let $\tilde{A}$ a p-adic lift of A and let $\tilde{A}_\eta$ be a p-adic lift of $\mathrm{Frac}(A)$ equipped with a morphism $\tilde{A} \hookrightarrow \tilde{A}_\eta$ lifting the inclusion $A \subseteq \mathrm{Frac}(A)$. We write S for the set of prime ideals $\mathfrak{p}$ of $\tilde{A}$ of codimension 1 containing p and for $\mathfrak{p} \in S$ we write $\tilde{A}_\mathfrak{p} \subseteq \tilde{A}_\eta$ for the p-adic completion of the localisation of $\tilde{A}$ at $\mathfrak{p}$. By construction, we have that $\tilde{A}_\mathfrak{p}/p = A_\mathfrak{p}$. This implies that $\bigcup_{\mathfrak{p}\in S} \tilde{A}_\mathfrak{p} \subseteq \tilde{A}_\eta$ is dense with respect to the p-adic topology since $\bigcup_{\mathfrak{p}\in S} A_\mathfrak{p} = \mathrm{Frac}(A)$.

We first want to prove that ring $\tilde{B} := \bigcap_{\mathfrak{p}\in S} \tilde{A}_\mathfrak{p}$ is equal to $\tilde{A}$. To do this, we first note that $\tilde{B}$ is p-adically complete and p-torsion-free since each $\tilde{A}_\mathfrak{p}$ is so. In addition, by the p-torsion-freeness, we have that $p\tilde{B} = \bigcap_{\mathfrak{p}\in S} p\tilde{A}_{\mathfrak{p}}$, which implies that the morphism $\tilde{B}/p \to \bigcap_{\mathfrak{p}\in S} \tilde{A}_{\mathfrak{p}}/p \subseteq \tilde{A}_\eta/p$ is injective. On the other hand, thanks to the algebraic Hartogs' lemma, we have that $\tilde{A}/p = \bigcap_{\mathfrak{p}\in S} \tilde{A}_{\mathfrak{p}}/p$. This implies that $\tilde{A}/p = \tilde{B}/p$ and in turn this shows that $\tilde{A} = \tilde{B}$.

Now, write M for the module over $\tilde{A}[\frac{1}{p}]$ associated to $\mathcal{M}$ and for every $\mathfrak{p} \in S$ write $M_{\mathfrak{p}}$ for the extension of scalars to $\tilde{A}_{\mathfrak{p}}[\frac{1}{p}]$. By [Dri22, Prop. 3.5.2], we have that M is a direct summand of $\tilde{A}[\frac{1}{p}]^{\oplus n}$ for some $n > 0$. Combining this with the fact that $\tilde{A}[\frac{1}{p}] = \bigcap_{\mathfrak{p}\in S} \tilde{A}_{\mathfrak{p}}[\frac{1}{p}]$, we deduce that

$$M = \bigcap_{\mathfrak{p}\in S} M_{\mathfrak{p}}. \tag{3.2}$$

By Lemma 3.3.3 applied to the inclusion $A \subseteq \mathrm{Frac}(A)$, if we denote by M_η the $\tilde{A}_\eta[\frac{1}{p}]$-module associated to $\mathcal{M}_\eta$, we get the cartesian square

$$\begin{tikzcd} H^0(\mathrm{Spec}(A), \mathcal{M}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & M \arrow[d] \\ H^0(\eta, \mathcal{M}_\eta) \arrow[r] & M_\eta. \end{tikzcd} \tag{3.3}$$

It remains to prove that every section $v \in H^0(\eta, \mathcal{M}_\eta)$ is also contained in M. By (3.2), this is equivalent to showing that v is in $M_{\mathfrak{p}}$ for every $\mathfrak{p} \in S$, which follows from Corollary 3.3.2. $\square$

Remark 3.3.5. Theorem 3.3.4 improves, a theorem by Kedlaya (see [DK17, Thm. 2.2.3]). We also prove in Theorem 5.2.1 a stronger form of Theorem 3.3.4 under the additional assumption that $\mathcal{M}$ upgrades to an F-isocrystal with slope filtration.

Theorem 3.3.4 can be used to prove the following result on the existence of the slope filtration.

Theorem 3.3.6 ([DvH22, Cor. 3.2.6])**.** *Let X be an irreducible Noetherian Frobenius-smooth scheme over $\mathbb{F}_p$. If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is an F-isocrystal over X with constant Newton polygon, then it admits the slope filtration.*

Proof. As in [Kat79], by taking exterior powers, it is enough to prove that if $(\mathcal{M}, \Phi_{\mathcal{M}})$ has minimal slope of multiplicity 1, then there exists a rank 1 sub-F-isocrystal of $(\mathcal{M}, \Phi_{\mathcal{M}})$ of minimal slope. Note also that the result is known on the generic point η of X (see [*ibid.*] and [dJO00, Claim 2.8]). If $S_1(\mathcal{M}_\eta) \subseteq \mathcal{M}_\eta$ is the subobject of minimal slope, up to taking a power of the Frobenius structure for some $s > 0$ and a Tate twist, we may assume that it corresponds to a lisse $\mathbb{Q}_{q^s}$-sheaf $\mathcal{F}_\eta$ over η. This lisse sheaf admits models over every codimension 1 point by [dJO00, Prop. 2.10], thus it admits an extension to a lisse $\mathbb{Q}_{q^s}$-sheaf $\mathcal{F}$ over X by Zariski–Nagata purity theorem. The lisse sheaf $\mathcal{F}$ corresponds then to an F^s-isocrystal $(\mathcal{N}, \Phi_{\mathcal{N}})$ over X which, by Theorem 3.3.4, embeds in $(\mathcal{M}, \Phi^s_{\mathcal{M}})$ providing a model of the inclusion $S_1(\mathcal{M}_\eta) \subseteq \mathcal{M}_\eta$. This yields the desired result. $\square$

Remark 3.3.7. Note that in [Ked24] Kedlaya proves the analogue of Theorem 3.3.6 for perfect schemes using arc-descent.

3.4. **Dieudonné modules of p-divisible groups.** One of the key aspects of crystalline cohomology, which motivated the very construction, is its relationship with *p-divisible groups*. Let S be a scheme of positive characteristic p.

Definition 3.4.1. A *p-divisible group* over S is a sheaf $\mathbb{X}$ of abelian groups over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ such that the following conditions are satisfied.

(1) $\mathbb{X} = \varinjlim_n \mathbb{X}[p^n]$ (p-torsion).
(2) $\cdot p : \mathbb{X} \to \mathbb{X}$ is surjective (p-divisible).
(3) $\mathbb{X}[p^n]$ is a finite locally-free group scheme for every[5] $n \geq 0$.

Example 3.4.2. Basic examples of p-divisible groups are $\mathbb{Q}_p/\mathbb{Z}_p \coloneqq \varinjlim_n \mathbb{Z}/p^n$ and $\mu_{p^\infty} \coloneqq \varinjlim_n \mu_{p^n}$. A reacher source of examples is provided by the p-divisible group attached to a commutative abelian scheme $\mathcal{A}$ over S. This is defined by $\mathcal{A}[p^\infty] \coloneqq \varinjlim_n \mathcal{A}[p^n]$.

Given a p-divisible group $\mathbb{X}$, we also have a notion of *Cartier dual*

$$\mathbb{X}^\vee \coloneqq \varinjlim_n \mathcal{H}\mathrm{om}(\mathbb{X}[p^n], \mathbb{G}_{m,S}),$$

of *Tate module*

$$T_p\mathbb{X} \coloneqq \varprojlim_{\cdot p} \mathbb{X}[p^n] = \mathcal{H}\mathrm{om}(\mathbb{Q}_p/\mathbb{Z}_p, \mathbb{X}),$$

and *universal cover*

$$\tilde{\mathbb{X}} \coloneqq \varprojlim_{\cdot p} \mathbb{X} = \mathcal{H}\mathrm{om}(\mathbb{Q}_p/\mathbb{Z}_p, \mathbb{X})[\tfrac{1}{p}].$$

These sheaves satisfy the fpqc sheaf condition as well. Note also that $T_p\mathbb{X}$ is a sheaf of $\mathbb{Z}_p$-modules while $\tilde{\mathbb{X}}$ is a sheaf of $\mathbb{Q}_p$-vector spaces.

Lemma 3.4.3 ([SW13, Prop. 3.3.1])**.** *If $\mathbb{X}$ is connected, then $T_p\mathbb{X}$, and $\tilde{\mathbb{X}}$ sit in the following exact sequence as fpqc sheaves*

$$0 \to T_p\mathbb{X} \to \tilde{\mathbb{X}} \to \mathbb{X} \to 0. \tag{3.4}$$

Remark 3.4.4. Note that (3.4) fails to be exact for the fppf topology since $\tilde{\mathbb{X}} \to \mathbb{X}$ fails to be surjective.

We also recall that $\mathbb{X}^\circ \coloneqq \varinjlim_n \mathbb{X}[p^n]^\circ$ is a p-divisible subgroup of $\mathbb{X}$ and if S is the spectrum of a field we have an exact sequence

$$0 \to \mathbb{X}^\circ \to \mathbb{X} \to \mathbb{X}^{\text{ét}} \to 0,$$

where $\mathbb{X}^{\text{ét}} \coloneqq \varinjlim_n \mathbb{X}[p^n]^{\text{ét}}$. This should be thought as a coarser version of the slope filtration.

Let us recall the following fundamental representability results for the objects we defined.

Theorem 3.4.5 ([Mes72])**.** *If $\mathbb{X}$ is connected, then $\mathbb{X} \to S$ is Zariski-locally represented by*

$$\mathrm{Spf}(R[[M]]) \to \mathrm{Spec}(R)$$

with M a finite free R-module.

[5]It is actually sufficient to ask for this condition when $n = 1$.

Corollary 3.4.6 ([SW13, Prop. 3.1.3, Prop. 3.3.1])**.** *If S is perfect and $\mathbb{X}$ is connected, then Zariski-locally $\tilde{\mathbb{X}} \to S$ is represented by*

$$\mathrm{Spf}(R[[x_1^{1/p^\infty}, \cdots, x_d^{1/p^\infty}]]) \to \mathrm{Spec}(R)$$

while $T_p\mathbb{X} \to S$ is represented by

$$\mathrm{Spec}(R[x_1^{1/p^\infty}, \cdots, x_d^{1/p^\infty}]/(x_1, \cdots, x_d)) \to \mathrm{Spec}(R).$$

Connected p-divisible groups should be thought then as commutative groups that are geometrically p-adic fibrations in open balls. Following the original idea of Dieudonné, it has been developed a linearisation procedure that captures the behaviour of these fibrations and their commutative group structures. This can be now expressed using the crystalline site of S and reminds the Lie algebra construction associated to a Lie group. This linearisation often provide a complete classification of p-divisible groups, analogous to how Lie algebras classify simply connected Lie groups.

Definition 3.4.7. A *Dieudonné module* over S is a locally-free coherent crystal $\mathcal{M}^+$ over S endowed with an isomorphism $\Phi_{\mathcal{M}^+} \colon F^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$ such that

$$\mathcal{M}^+ \subseteq \Phi_{\mathcal{M}^+}(F^*\mathcal{M}^+) \subseteq \tfrac{1}{p}\mathcal{M}^+.$$

We write $\mathbf{DM}(S)$ for the category they form.

Theorem 3.4.8 ([BBM82],[deJ95, Thm. 4.1.1])**.** *If S is a Frobenius-smooth scheme there exists an equivalence of categories*

$$\mathbb{D} \colon \{\textit{p-divisible groups over } S\} \to \mathbf{DM}(S),$$

called crystalline Dieudonné module functor.

3.5. **Quasi-regular semiperfect rings.** Another nice class of schemes where crystals have a particularly good behaviour are the affine schemes which are the spectrum of a *quasi-regular semiperfect ring*.

Definition 3.5.1. We say that a scheme S over $\mathbb{F}_p$ is *quasi-syntomic* if its cotangent complex $\mathbb{L}_{S/\mathbb{F}_p}$ has Tor-amplitude in the interval $[-1, 0]$. We also say that an fpqc cover $T \to S$ is a *quasi-syntomic cover* if $\mathbb{L}_{T/S}$ has Tor-amplitude in the interval $[-1, 0]$. Note that by [Stacks, Tag 0FJV], an fppf cover $T \to S$ is syntomic if and only if it is quasi-syntomic (hence the name). If $S = \mathrm{Spec}(R)$ is quasi-syntomic and R is semiperfect, then we say that R is *quasi-regular semiperfect* (qrsp).

The notion of quasi-regular semiperfect rings is related to another notion introduced by Quillen.

Definition 3.5.2. An ideal I of a ring A is said to be *quasi-regular* if the cotangent complex $\mathbb{L}_{(A/I)/A}$ has Tor-amplitude concentrated in degree -1.

The relation between the two definitions is explained by the following lemma.

Lemma 3.5.3. *Let A be a perfect ring and $A \twoheadrightarrow R$ a quotient with kernel I, then R is qrsp if and only if I is quasi-regular*

Proof. By functoriality, the absolute Frobenius of A induces on $\mathbb{L}_{A/\mathbb{F}_p}$ an automorphism. This has to be the 0-morphism since, in characteristic p, the differential forms dx^p vanish. This implies that $\mathbb{L}_{A/\mathbb{F}_p} = 0$. Thanks to the base change triangle for the cotangent complex, we deduce that $\mathbb{L}_{R/\mathbb{F}_p} \xrightarrow{\sim} \mathbb{L}_{R/A}$. It remains to prove that if R is quasi-syntomic, then $\pi_0(\mathbb{L}_{R/A}) = 0$. This follows from the fact that, since $A \to R$ is surjective, the module $\Omega^1_{R/A}$ vanishes. $\square$

Example 3.5.4. The main example of qrsp rings, besides the perfect ones, are the rings of the form

$$\mathrm{Spec}(R[x_1^{1/p^\infty}, \cdots, x_d^{1/p^\infty}]/(x_1, \cdots, x_d))$$

with R a perfect ring. Thus, by Proposition 3.4.6, an example is provided by the algebra of functions of the Tate module of a connected p-divisible group over a perfect field. We will see that the stabilisers of the formal homogeneous spaces of §6.3 are of this form.

Suppose that R is qrsp[6] and write $S := \mathrm{Spec}(R)$. As noted by Fontaine, the crystalline site $\mathrm{Cris}(S/\Sigma_n)$ admits a final object for every $n \geq 0$, as in the case of perfect fields. Let us briefly recall the construction.

Let $R^\flat$ be the perfect ring

$$\lim\, (\cdots \xrightarrow{\varphi} R \xrightarrow{\varphi} R)$$

and let J be the kernel of the composition

$$W(R^\flat) \to R^\flat \to R^\flat/p.$$

We write[7] $A_{\mathrm{cris}}(R) \subseteq W(R^\flat)[\frac{1}{p}]$ for the smallest p-adically complete subring containing $W(R^\flat)$ and $\frac{x^n}{n!}$ for all $x \in J$ and $n \geq 0$. By construction, $JA_{\mathrm{cris}}(R)$ admits a canonical PD-structure induced by the one of $JW(R^\flat)[\frac{1}{p}]$. One can check that for every n the scheme $\mathrm{Spec}(A_{\mathrm{cris}}(R)/p^n)$ is the final object of $\mathrm{Cris}(S/\Sigma_n)$. As a consequence, the category of locally-free coherent crystals over S is equivalent to the category of locally finitely-generated and free $A_{\mathrm{cris}}(R)$-modules.

Theorem 3.5.5 ([ALB23, Thm. 4.8.5])**.** *If S is the spectrum of a qrsp ring, there exists a fully faithful functor*

$$\mathbb{D}\colon \{p\text{-divisible groups over } S\} \to \mathbf{DM}(S),$$

which is functorial in S.

Corollary 3.5.6. *If $\mathbb{X}$ is a p-divisible group over a qrsp ring R, we have the following natural commutative diagram*

$$\begin{tikzcd}
T_p\mathbb{X}(R) \arrow[r, "="] \arrow[d, hook] & \mathbb{D}(\mathbb{X})^{\varphi=\mathrm{id}} \arrow[d, hook] \\
\tilde{\mathbb{X}}(R) \arrow[r, "="] & (\mathbb{D}(\mathbb{X})[\frac{1}{p}])^{\varphi=\mathrm{id}}.
\end{tikzcd}$$

[6]Some of the things we will say here are actually true for general semiperfect rings.

[7]We give here a slightly *ad hoc* definition of $A_{\mathrm{cris}}(R)$ that coincides with the standard one since R is qrsp, see [SW13, Prop. 4.1.11]. In general, one should rather use the notion of *PD-envelope*.

Proof. We have

$$\begin{aligned} T_p\mathbb{X}(R) &= \mathrm{Hom}_R((\mathbb{Q}_p/\mathbb{Z}_p)_R, \mathbb{X}) \\ &= \mathrm{Hom}_{A_{\mathrm{cris}},\varphi}(A_{\mathrm{cris}}(R), \mathbb{D}(\mathbb{X})) \\ &= \mathbb{D}(\mathbb{X})^{\varphi=1}. \end{aligned}$$

We can argue similarly after inverting p to get the analogous result for $\tilde{\mathbb{X}}(R)$. □

3.6. **Complete slope divisibility.** We conclude this section with some recalls on the notion of complete slope divisibility.

Definition 3.6.1. For a perfect ring R we say that an isoclinic Dieudonné module (M^+, φ_{M^+}) over R is *completely slope divisible* if there exist integers s and a with $s \neq 0$ such that $\varphi^s_{M^+}M^+ = p^a M^+$. We also say that a Dieudonné module (M^+, φ_{M^+}) over R is *completely slope divisible* if it is the direct sum of isoclinic completely slope divisible Dieudonné modules and we say that a p-divisible group is *completely slope divisible* if the associated Dieudonné module is so.

Remark 3.6.2. Since we assumed that R is perfect, the definition we gave is equivalent to the usual one thanks to [OZ02, Prop. 1.3]. Note also that by [*ibid.*, Cor. 1.5], if R is an algebraically closed field, an isoclinic Dieudonné module is completely slope divisible if and only if it is defined over a finite field.

Lemma 3.6.3. *A Dieudonné module over $\overline{\mathbb{F}}_p$ is completely slope divisible if and only if it is a direct sum of isoclinic Dieudonné modules.*

Proof. By [OZ02, Cor. 1.5] it is enough to prove that every isoclinic Dieudonné module is defined over a finite field. By Dieudonné theory, this follows from the fact that a p-divisible group which is geometrically isogenous to a p-divisible group defined over a finite field, is itself defined over a finite field. □

Lemma 3.6.4. *If (M^+, φ_{M^+}) is a completely slope divisible Dieudonné module over $\overline{\mathbb{F}}_p$ and (N^+, φ_{N^+}) is a Dieudonné submodule such that M^+/N^+ is torsion-free, then (N^+, φ_{N^+}) is completely slope divisible.*

Proof. By Lemma 3.6.3, we have to prove that (N^+, φ_{N^+}) is a direct sum of isoclinic Dieudonné modules. By the Dieudonné–Manin classification, there exists a Dieudonné submodule $(\bar{N}^+, \varphi_{\bar{N}^+}) \subseteq (N^+, \varphi_{N^+})$ of finite index which decomposes into a direct sum $\oplus_{\lambda\in\mathbb{Q}}\bar{N}^+_\lambda$ of isoclinic Dieudonné modules. For an element $x \in N^+$, there exist by assumption $x_\lambda \in M^+_\lambda$ for $\lambda \in \mathbb{Q}$ almost all 0 such that $x = \sum_{\lambda\in\mathbb{Q}} x_\lambda$. Since $\bar{N}^+ \subseteq N^+$ is of finite index, there exists n big enough such that $p^n x \in \bar{N}^+$, so that $p^n x_\lambda \in \bar{N}^+$ for every λ. This implies that $p^n x_\lambda \in N^+$ for every λ and by the assumption that M^+/N^+ is torsion-free, we deduce that each x_λ lies in N^+. This yields the desired result. □

4. Parabolicity conjecture

4.1. **Introduction.** Let X be a smooth geometrically connected variety over a perfect field k of positive characteristic p. For an overconvergent F^n-isocrystal $(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}})$ over X we write $(\mathcal{M}, \Phi_{\mathcal{M}})$ for the associated F-isocrystal and we suppose that $(\mathcal{M}, \Phi_{\mathcal{M}})$ admits the slope filtration

$$0 = S_0(\mathcal{M}) \subsetneq S_1(\mathcal{M}) \subsetneq ... \subsetneq S_m(\mathcal{M}) = \mathcal{M}.$$

Let η be a point of X with perfect residue field and consider the monodromy groups $G(\mathcal{M},\eta)$ and $G(\mathcal{M}^\dagger,\eta)$. The former algebraic group is a subgroup of the latter and they both are subgroups of $\mathrm{GL}(\mathcal{M}_\eta)$, where $\mathcal{M}_\eta$ is the fibre of $\mathcal{M}$ at η. In [D'Ad23] we proved the following fundamental result about these groups.

Theorem 4.1.1. *The subgroup $G(\mathcal{M},\eta) \subseteq G(\mathcal{M}^\dagger,\eta)$ is the subgroup of $G(\mathcal{M}^\dagger,\eta)$ stabilising the slope filtration of $\mathcal{M}_\eta$. Moreover, when $\mathcal{M}^\dagger$ is semi-simple, $G(\mathcal{M},\eta)$ is a parabolic subgroup of $G(\mathcal{M}^\dagger,\eta)$.*

This solved the *parabolicity conjecture*, initially proposed in [Cre92a, page 460]. Partial results on this conjecture were previously obtained in [Cre92b], [Cre94], [Tsu02], [AD22], and [Tsu23]. Theorem 4.1.1 can be seen as a natural enhancement of Kedlaya's full faithfulness theorem, proven in [Ked04b].

When k is a finite field and $\mathcal{M}^\dagger$ is semi-simple, we established in [D'Ad20] a fundamental comparison theorem for the group $G(\mathcal{M}^\dagger,\eta)^\circ$ and the corresponding monodromy group of the semi-simple ℓ-adic lisse sheaves with the same L-function as $(\mathcal{M}^\dagger,\Phi^\dagger_{\mathcal{M}})$ (see also [Dri18] and [Pál22]). Since the monodromy groups of lisse sheaves are much better understood,this comparison allows us to compute $G(\mathcal{M}^\dagger,\eta)^\circ$ in many cases where it was not known. This comparison theorem is a highly nontrivial result which uses the Langlands correspondence for lisse sheaves and overconvergent F-isocrystals, [Laf02], [Abe18]. By combining [D'Ad20] and Theorem 4.1.1, one can compute the group $G(\mathcal{M},\eta)$.

As first noted by Chai, the determination of $G(\mathcal{M},\eta)$ in the case of Shimura varieties could have been used to attack Chai–Oort Hecke orbit conjecture (see [Cha13, §7]). After the resolution of the parabolcity conjecture, this insight indeed led to a proof of Chai–Oort's conjecture. The conjecture was first proved by van Hoften in the ordinary case [vH24], and was subsequently generalised in [DvH22]. In §6, we will present some ideas of the proof. Beyond the Hecke orbit conjecture, the parabolicity conjecture has emerged as a powerful tool in the study of Shimura varieties modulo p, as explored in [vHX21] and [Jia23].

Another consequence of Theorem 4.1.1, is the following result, which was a conjecture proposed by Kedlaya in [Ked22, Rmk. 5.14].

Theorem 4.1.2 ([D'Ad23, Cor. 1.1.4])**.** *Let X be a smooth connected variety over a perfect field k and let $(\mathcal{M}_1^\dagger,\Phi^\dagger_{\mathcal{M}_1})$ and $(\mathcal{M}_2^\dagger,\Phi^\dagger_{\mathcal{M}_2})$ be two irreducible overconvergent F^n-isocrystals over X with constant slopes. If $(S_1(\mathcal{M}_1),\Phi_{\mathcal{M}_1}|_{S_1(\mathcal{M}_1)})$ and $(S_1(\mathcal{M}_2),\Phi_{\mathcal{M}_2}|_{S_1(\mathcal{M}_2)})$ are isomorphic F^n-isocrystals, then $(\mathcal{M}_1^\dagger,\Phi^\dagger_{\mathcal{M}_1})$ and $(\mathcal{M}_2^\dagger,\Phi^\dagger_{\mathcal{M}_2})$ are isomorphic overconvergent F^n-isocrystals.*

The conjecture was solved in dimension 1 and when k is a finite field by Tsuzuki in [Tsu23].

One of the main tools introduced and studied in [D'Ad23] is the notion of $\dagger$-*hull* of a sub-F^n-isocrystal.

Definition 4.1.3. Let $(\mathcal{N},\Phi_{\mathcal{N}}) \subseteq (\mathcal{M},\Phi_{\mathcal{M}})$ be an inclusion of F^n-isocrystals. The $\dagger$-*hull* of $(\mathcal{N},\Phi_{\mathcal{N}})$ in $(\mathcal{M},\Phi_{\mathcal{M}})$ is the smallest subobject of $(\mathcal{M},\Phi_{\mathcal{M}})$ containing $(\mathcal{N},\Phi_{\mathcal{N}})$ and coming from an overconvergent F^n-isocrystal. We denote it by $(\overline{\mathcal{N}},\Phi_{\overline{\mathcal{N}}})$.

We obtained the following fundamental result which relates the slope filtration with the operation of taking †-hull7s.

Theorem 4.1.4 (Theorem 4.3.2)**.** *Let X be a smooth variety over a perfect field k and let $(\mathcal{N}, \Phi_{\mathcal{N}}) \subseteq (\mathcal{M}, \Phi_{\mathcal{M}})$ be an inclusion of F^n-isocrystals over X. If $\mathcal{M}$ comes from an overconvergent isocrystal and has constant slopes, then $S_1(\mathcal{N}, \Phi_{\mathcal{N}}) = S_1(\overline{\mathcal{N}}, \Phi_{\overline{\mathcal{N}}})$.*

When X is a curve, we first reduced the statement to the case $X = \mathbb{A}^1_k$, then we proved an analogous result for the generic point of $\mathbb{A}^1_k$ using de Jong's *reverse filtratio*n (Theorem 4.3.9). Finally, we deduced the global result from the generic one. The idea of using Theorem 4.3.9 was suggested to the author by Kedlaya and independently used in [Tsu23, Prop. 6.1]. In his proof Tsuzuki makes use also of a certain filtration for overconvergent F-isocrystals, namely the PBQ *filtration*, that he constructs in [Tsu23, Thm. 3.27]. Our proof avoids the use of this filtration and uses instead some ideas of what is now called *edged crystalline cohomology*, [D'Ad24b]. At the same time, with Theorem 4.1.4 we recovered the PBQ *filtration* and we extended it to arbitrary smooth varieties, [D'Ad23, Cor. 5.4.2].

To deduce Theorem 4.1.1 from Theorem 4.1.4, we proved first that Theorem 4.1.4 implies the analogue of Theorem 4.1.1 for some slightly different groups: the monodromy groups of F^∞-*isocrystals* (Proposition 4.4.2). Subsequently, to pass from this variant to the original statement, we introduced a third type of monodromy groups: the monodromy groups of *isocrystals with punctual $\mathbb{Q}_p^{\mathrm{ur}}$-structure*, defined in §4.2.8. These latter monodromy groups are $\mathbb{Q}_p^{\mathrm{ur}}$-forms of Crew's monodromy groups (the ones defined in [Cre92a]), thus we are then able to prove Theorem 4.1.1.

For higher dimensional varieties we deduced Theorem 4.1.4 from the case of curves, thanks to a new Lefschetz theorem for overconvergent F^n-isocrystals (Theorem 4.5.1).

4.2. **Punctual $\mathbb{Q}_p^{\mathrm{ur}}$-structures.** In the proof of the parabolicity conjecture we had to deal with the fact that the field of scalars of the category of isocrystals is in general much bigger than the field of scalars of the category of F-isocrystals. We encountered this issue when we wanted to prove that Theorem 4.1.4 implies Theorem 4.1.1. This implication, by its nature, is easier for the monodromy groups of F^n-isocrystals, but it is much harder for the monodromy groups defined by Crew (without Frobenius structure). To jump from one setting to the other we slightly modified both categories. We first replaced the category of F^n-isocrystals with the category of F^∞-isocrystals, namely the 2-colimit of the categories of F^n-isocrystals for various n. If k is big enough, this new category is a $\mathbb{Q}_p^{\mathrm{ur}}$-linear category. On the other side, we constructed the category of isocrystals with punctual $\mathbb{Q}_p^{\mathrm{ur}}$-structure (see §4.2.2), which is simply the category of isocrystals endowed with the choice of a $\mathbb{Q}_p^{\mathrm{ur}}$-linear lattice at some fibre. This other Tannakian category is also $\mathbb{Q}_p^{\mathrm{ur}}$-linear.

There is a natural functor between these two categories thanks to Dieudonné–Manin classification (Lemma 4.2.5). To prove Theorem 4.1.1 we relate the monodromy groups of these objects and their overconvergent variants thanks to Proposition 4.2.9, which is an analogue of the homotopy exact sequence for the étale fundamental group. Let us explain this more in detail.

Hypothesis 4.2.1. Throughout this section, we assume that k is endowed with the choice of an inclusion $\overline{\mathbb{F}}_p \subseteq k$. In particular, for every perfect field extension Ω/k we have a preferred embedding $\mathbb{Q}_p^{\mathrm{ur}} \hookrightarrow K(\Omega)$.

Definition 4.2.2. If η is a geometric point of X, an *isocrystal with (punctual) $\mathbb{Q}_p^{ur}$-structure* over (X,η) is a pair $(\mathcal{M}, V_{\mathcal{M}})$ where $\mathcal{M}$ is an isocrystal and $V_{\mathcal{M}}$ is a $\mathbb{Q}_p^{\mathrm{ur}}$-linear lattice of $\omega_\eta(\mathcal{M})$, namely a $\mathbb{Q}_p^{\mathrm{ur}}$-linear vector subspace $V_{\mathcal{M}} \subseteq \omega_\eta(\mathcal{M})$ such that

$$V_{\mathcal{M}} \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} K(\Omega) = \omega_\eta(\mathcal{M}).$$

A morphism of isocrystals with $\mathbb{Q}_p^{\mathrm{ur}}$-structure $(\mathcal{M}, V_{\mathcal{M}}) \to (\mathcal{N}, V_{\mathcal{N}})$ is a morphism of isocrystals $f : \mathcal{M} \to \mathcal{N}$ such that $\omega_\eta(f)(V_{\mathcal{M}}) \subseteq V_{\mathcal{N}}$. We write $\mathbf{Isoc}_{\mathbb{Q}_p^{\mathrm{ur}}}(X,\eta)$ for the category of isocrystals with punctual $\mathbb{Q}_p^{\mathrm{ur}}$-structure over (X,η).

Lemma 4.2.3. *If X is geometrically connected, the category $\mathbf{Isoc}_{\mathbb{Q}_p^{ur}}(X,\eta)$ has a natural structure of a $\mathbb{Q}_p^{ur}$-linear neutral Tannakian category.*

Now that we have a neutral Tannakian category of isocrystals with smaller field of scalars, we want to study its interaction with the category of F^n-isocrystals.

Construction 4.2.4. Let Ω/k be an algebraically closed field extension. For an F^n-isocrystal (M, Φ_M) over Ω, we write $\omega_{\mathbb{Q}_p^{\mathrm{ur}}}(M, \Phi_M) \subseteq M$ for the $\mathbb{Q}_p^{\mathrm{ur}}$-linear vector subspace of vectors $v \in M$ such that $\Phi_M^i v = p^j v$ for some $(i,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}$.

Lemma 4.2.5. *The vector space $\omega_{\mathbb{Q}_p^{ur}}(M, \Phi_M)$ is a $\mathbb{Q}_p^{ur}$-linear lattice of the $K(\Omega)$-vector space M.*

Proof. This is an immediate consequence of Theorem 3.1.4. □

Definition 4.2.6. We say that $\omega_{\mathbb{Q}_p^{\mathrm{ur}}}(M, \Phi_M)$ is the *Dieudonné–Manin $\mathbb{Q}_p^{ur}$-structure* of (M, Φ_M). The assignment $(M, \Phi_M) \mapsto \omega_{\mathbb{Q}_p^{\mathrm{ur}}}(M, \Phi_M)$ produces a $\mathbb{Q}_p^{\mathrm{ur}}$-linear fibre functor $\omega_{\mathbb{Q}_p^{\mathrm{ur}}} : \mathbf{F^\infty\text{-}Isoc}(\Omega) \to \mathbf{Vec}_{\mathbb{Q}_p^{\mathrm{ur}}}$. If η is a Ω-point of X, we write

$$\omega_{\eta,\mathbb{Q}_p^{\mathrm{ur}}} : \mathbf{F^\infty\text{-}Isoc}(X) \to \mathbf{Vec}_{\mathbb{Q}_p^{\mathrm{ur}}}$$

for the composition $\omega_{\mathbb{Q}_p^{\mathrm{ur}}} \circ \eta^*$ and we write

$$\Lambda_\eta : \mathbf{F^\infty\text{-}Isoc}(X) \to \mathbf{Isoc}_{\mathbb{Q}_p^{\mathrm{ur}}}(X,\eta)$$

for the functor obtained by sending $(\mathcal{M}, \Phi_{\mathcal{M}}^\infty) \mapsto (\mathcal{M}, \omega_{\eta,\mathbb{Q}_p^{\mathrm{ur}}}(\mathcal{M}, \Phi_{\mathcal{M}}^\infty))$. We say that an object in the essential image of Λ_η is an *isocrystal with Dieudonné–Manin $\mathbb{Q}_p^{ur}$-structure* over (X,η).

Remark 4.2.7. The existence of the Dieudonné–Manin $\mathbb{Q}_p^{\mathrm{ur}}$-structure of an F-isocrystal over an algebraically closed field has its own interest. For example, thanks to [Ked06a], if k is any field of characteristic p, one can associate to a variety X/k the finite-dimensional $\mathbb{Q}_p^{\mathrm{ur}}$-linear vector spaces $\omega_{\mathbb{Q}_p^{\mathrm{ur}}}(H^i_{\mathrm{rig}}(X_{k^{\mathrm{alg}}}/K(k^{\mathrm{alg}})))$ and their variant with compact support. This assignment produces a $\mathbb{Q}_p^{\mathrm{ur}}$-linear cohomology theory with all the desired properties (e.g. Poincaré duality, the Künneth formula, etc.). This solves in a minimal way Serre's obstruction to the existence of a $\mathbb{Q}_p$-linear cohomology theory in characteristic p.

Definition 4.2.8. Let (X,η) be a geometrically connected variety over k endowed with a geometric point η. For an F^n-isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ we write $G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta)$ for the Tannaka group of $\langle \mathcal{M}, \Phi_{\mathcal{M}}^\infty \rangle$ with respect to $\omega_{\eta,\mathbb{Q}_p^{\mathrm{ur}}}$ and $G(\mathcal{M}, V_{\mathcal{M}}, \eta)$ for the Tannaka group of $\langle \mathcal{M}, V_{\mathcal{M}} \rangle$ with respect to $\omega_{\eta,\mathbb{Q}_p^{\mathrm{ur}}}$, where $V_{\mathcal{M}}$ is the Dieudonné–Manin $\mathbb{Q}_p^{\mathrm{ur}}$-structure induced by $\Phi_{\mathcal{M}}$. We also denote by $G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta)^{\mathrm{cst}}$ the Tannaka group of *constant* F^∞-isocrystals in $\langle \mathcal{M}, \Phi_{\mathcal{M}}^\infty \rangle$, namely those F^∞-isocrystals coming from $\mathrm{Spec}(k)$. We give analogous definitions in the overconvergent setting.

The fundamental result for these monodromy groups is the following exact sequence, reminiscence of the homotopy exact sequence for the étale fundamental group.

Proposition 4.2.9 ([D'Ad23, Prop. 3.2.8])**.** *For a †-extendable F^n-isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$, we have the following commutative diagram of algebraic groups over $\mathbb{Q}_p^{ur}$*

$$\begin{tikzcd}
1 \arrow[r] & G(\mathcal{M}, V_{\mathcal{M}}, \eta) \arrow[r] \arrow[d, hook] & G(\mathcal{M}, \Phi_{\mathcal{M}}^{\infty}, \eta) \arrow[r] \arrow[d, hook] & G(\mathcal{M}, \Phi_{\mathcal{M}}^{\infty}, \eta)^{\mathrm{cst}} \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & G(\mathcal{M}^{\dagger}, V_{\mathcal{M}}, \eta) \arrow[r] & G(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}}^{\dagger,\infty}, \eta) \arrow[r] & G(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}}^{\dagger,\infty}, \eta)^{\mathrm{cst}} \arrow[r] & 1.
\end{tikzcd}$$

The rows are exact, the first two vertical arrows are closed embeddings and the last arrow is a faithfully flat morphism.

Remark 4.2.10. Thanks to Theorem 4.1.4 one can also show that the morphism $G(\mathcal{M}, \Phi_{\mathcal{M}}^{\infty}, \eta)^{\mathrm{cst}} \to G(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}}^{\dagger,\infty}, \eta)^{\mathrm{cst}}$ is an isomorphism.

4.2.1. Another crucial analysis we had to undertake in [D'Ad23, §3] involved comparing the monodromy groups in $\mathbf{Isoc}_{\mathbb{Q}_p^{\mathrm{ur}}}(X, \eta)$ and $\mathbf{Isoc}(X)$. Generally, it is easy to construct examples where the scalar extension of the monodromy group of an isocrystal with $\mathbb{Q}_p^{\mathrm{ur}}$-structure from $\mathbb{Q}_p^{\mathrm{ur}}$ to K results in a larger monodromy group than the monodromy group of the isocrystal. However, for isocrystals with Dieudonné–Manin $\mathbb{Q}_p^{\mathrm{ur}}$-structure, we proved that the expected base change property holds.

Proposition 4.2.11 ([D'Ad23, Prop. 3.3.2])**.** *For an F^n-isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$, if $V_{\mathcal{M}}$ is the associated Dieudonné–Manin $\mathbb{Q}_p^{ur}$-structure, we have $G(\mathcal{M}, \eta) \simeq G(\mathcal{M}, V_{\mathcal{M}}, \eta) \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} K$.*

4.3. **The case of curves.**

Notation 4.3.1. Let $(\mathcal{M}, \Phi_{\mathcal{M}})$ be a †-extendable F^n-isocrystal over X with constant slopes. We say that $(\mathcal{M}, \Phi_{\mathcal{M}})$ satisfies $\mathrm{MS}(\mathcal{M}, \Phi_{\mathcal{M}})$ (where MS stands for "minimal slope") if for every sub-F^n-isocrystal $(\mathcal{N}, \Phi_{\mathcal{N}}) \subseteq (\mathcal{M}, \Phi_{\mathcal{M}})$, the isocrystals $S_1(\mathcal{N})$ and $S_1(\overline{\mathcal{N}})$ are the same. We also say that X satisfies $\mathrm{MS}(X)$ if for every $n > 0$ and every †-extendable F^n-isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ over X, we have that $\mathrm{MS}(\mathcal{M}, \Phi_{\mathcal{M}})$ is true.

Theorem 4.1.4 can be then written in the following form.

Theorem 4.3.2. *A smooth variety X over a perfect field k satisfies* $\mathrm{MS}(X)$*.*

4.3.1. At the beginning of [D'Ad23, §4] we proved some reductions of Theorem 4.3.2. We showed that we could assume $n = 1$, that we could check $\mathrm{MS}(X)$ on isoclinic sub-F-isocrystals, and we proved the following lemma.

Lemma 4.3.3. *Let $f: Y \to X$ be a dominant étale morphism between smooth varieties.*
 (1) $\mathrm{MS}(Y)$ *implies* $\mathrm{MS}(X)$*.*
 (2) *If f is finite, then* $\mathrm{MS}(X)$ *implies* $\mathrm{MS}(Y)$*.*

Combining Lemma 4.3.3 and [Ked05], in dimension 1 we reduced $\mathrm{MS}(X)$ to $\mathrm{MS}(\mathbb{A}_k^1)$. In turn, to prove $\mathrm{MS}(\mathbb{A}_k^1)$ we used an analogue of $\mathrm{MS}(-)$ for the generic point of $\mathbb{A}_k^1$.

4.3.2. Consider the ring $\mathcal{O}_{\mathcal{E}} := (W[t]_{(p)})^{\wedge}$, where $(-)^{\wedge}$ denotes the p-adic completion. This is a complete discrete valuation ring unramified over W with residue field $k(t)$. Let $\mathcal{O}_{\mathcal{E}^\dagger} \subseteq \mathcal{O}_{\mathcal{E}}$ be the subring of functions which converge in some annulus $* \leq |t| < 1$. These two rings are both endowed with a Frobenius lift $\varphi(t) = t^p$ and a derivation ∂_t. Write $\mathcal{E}$ and $\mathcal{E}^\dagger$ for the respective fields of fractions.

Definition 4.3.4 ((φ, ∇)-modules)**.** If E is either $\mathcal{E}$ or $\mathcal{E}^\dagger$, we say that a finite dimensional vector space M over E is a *(φ, ∇)-module* if it is endowed with a φ-linear isomorphism $\varphi_M : M \xrightarrow{\sim} M$ and an additive morphism $\nabla_{\partial_t} : M \to M$ which satisfies the Leibniz rule and such that $\nabla_{\partial_t} \circ \varphi_M = pt^{p-1}\varphi_M \circ \nabla_{\partial_t}$.

The category F-Isoc($k(t)$) is the category of (φ, ∇)-modules over $\mathcal{E}$ and we denote by F-Isoc†($k(t)$) the category of (φ, ∇)-modules over $\mathcal{E}^\dagger$. Thanks to Theorem 5.1 [Ked04b], the natural functor

$$\text{F-Isoc}^\dagger(k(t)) \to \text{F-Isoc}(k(t))$$

is fully faithful.

Proposition 4.3.5 (Kedlaya, Tsuzuki[8])**.** *If $N \subseteq M$ is an inclusion of (φ, ∇)-modules over $\mathcal{E}$ and M is $\dagger$-extendable, then $S_1(N) = S_1(\overline{N})$.*

To prove Proposition 4.3.5 we first need the following construction.

Construction 4.3.6. Let $Q^\dagger$ be the image of the composition of the natural morphisms

$$(M^\dagger)^\vee := \mathrm{Hom}_{\mathcal{E}^\dagger}(M^\dagger, \mathcal{E}^\dagger) \to \mathrm{Hom}_{\mathcal{E}}(M, \mathcal{E}) \to \mathrm{Hom}_{\mathcal{E}}(N, \mathcal{E}) =: N^\vee.$$

We have natural maps

$$M^\vee = (M^\dagger)^\vee \otimes_{\mathcal{E}^\dagger} \mathcal{E} \twoheadrightarrow Q^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{E} \twoheadrightarrow N^\vee.$$

The first arrow is surjective by construction, the second one is surjective because the morphism $M^\vee \to N^\vee$ is surjective. Note that even though $Q^\dagger \subseteq N^\vee$, the second map needs not to be injective. Dualising with respect to $\mathcal{E}$ we get inclusions $N \subseteq Q^\vee \subseteq M$.

Lemma 4.3.7. *The (φ, ∇)-module $Q^\vee$ is the $\dagger$-hull of N in M. In other words, $\overline{N}$ is the unique submodule of M which contains N and comes from some $\overline{N}^\dagger \subseteq M^\dagger$ such that $(\overline{N}^\dagger)^\vee \to N^\vee$ is injective.*

Proof. By construction, $Q^\vee$ is $\dagger$-extendable and it contains N, so that $\overline{N} \subseteq Q^\vee$. On the other hand, we have morphisms $(M^\dagger)^\vee \twoheadrightarrow (\overline{N}^\dagger)^\vee \to N^\vee$, where the first one is surjective. By definition, the morphism $(\overline{N}^\dagger)^\vee \to N^\vee$ factors through $Q^\dagger$, which implies that $Q^\vee \subseteq \overline{N}$. □

We recall now the reverse filtration introduced by de Jong in [deJ98, Prop. 5.5]. For this, we need to introduce two other discrete valuation fields lifting $\overline{k(t)}$.

[8]We first learned about a proof of Proposition 4.3.5 from Kedlaya via a private communication. The proposition also corresponds essentially to [Tsu23, Thm. 2.14].

Definition 4.3.8. Let $\mathcal{O}_{\mathcal{E}_{\mathrm{alg}}}$ be the ring of Witt vectors of $\overline{k(t)}$ (which is contained in the ring $\Gamma_2 = \Gamma_{2,1}$ in de Jong's notation). Every element of $\mathcal{O}_{\widetilde{\mathcal{E}}}$ can be written uniquely as $\sum_{i=0}^{\infty}[f_i]p^i$ where $[f_i]$ is the Teichmüller lift of some $f_i \in \overline{k(t)}$. Consider the subring $\mathcal{O}^{\dagger}_{\widetilde{\mathcal{E}}} \subseteq \mathcal{O}_{\widetilde{\mathcal{E}}}$ of those series such that the t-adic valuations of f_i are bounded below by some linear function in i. This subring is preserved by the Frobenius of $\mathcal{O}_{\widetilde{\mathcal{E}}}$ (and it is contained in $\Gamma_{2,c} = \Gamma_{2,1,c}$ in de Jong's notation). We write $\widetilde{\mathcal{E}}^{\dagger}$ and $\widetilde{\mathcal{E}}$ for the fraction fields.

Theorem 4.3.9 ([deJ98, Prop. 5.5])**.** *For a φ-module $M^{\dagger}_{\mathrm{alg}}$ over $\widetilde{\mathcal{E}}^{\dagger}$ the following statements are true.*

(i) *$M^{\dagger}_{\mathrm{alg}}$ admits a reverse slope filtration, i.e. there exists a filtration*
$$0 = S^{\mathrm{rev}}_0(M^{\dagger}_{\mathrm{alg}}) \subsetneq S^{\mathrm{rev}}_1(M^{\dagger}_{\mathrm{alg}}) \subsetneq \cdots \subsetneq S^{\mathrm{rev}}_m(M^{\dagger}_{\mathrm{alg}}) = M^{\dagger}_{\mathrm{alg}}$$
of φ-modules over $\widetilde{\mathcal{E}}^{\dagger}$ such that $(S^{\mathrm{rev}}_i(\widetilde{M}^{\dagger})/S^{\mathrm{rev}}_{i-1}(\widetilde{M}^{\dagger})) \otimes_{\widetilde{\mathcal{E}}^{\dagger}} \widetilde{\mathcal{E}}$ is isomorphic to $S_{m-i}(M_{\mathrm{alg}})/S_{m-i-1}(M_{\mathrm{alg}})$.

(ii) *If $M^{\dagger}$ is isoclinic of slope s/r, after possibly multiplying s and r by some positive integer, the φ-module $M^{\dagger}_{\mathrm{alg}}[p^{1/r}]$ admits a basis of vectors $\{v_1, \dots, v_d\}$ such that $\varphi(v_i) = p^{s/r}v_i$.*

Lemma 4.3.10. *Let $M^{\dagger}$ be a φ-module over $\mathcal{E}^{\dagger}$ and let N be an isoclinic φ-module over $\mathcal{E}$ of slope s/r. For every morphism $\psi : M \to N$ of φ-modules, if the restriction of ψ to $M^{\dagger}$ is injective, then the maximal slope of M is s/r and the rank of $S_m(M)/S_{m-1}(M)$ is smaller or equal than the rank of N.*

Proof. This is a variant of [Ked04b, Lem. 4.2]. Since $\mathcal{E}^{\dagger}_{\mathrm{alg}}$ is flat over $\mathcal{E}^{\dagger}$ and $\mathcal{E} \otimes_{\mathcal{E}^{\dagger}} \widetilde{\mathcal{E}}^{\dagger} \to \widetilde{\mathcal{E}}$ is injective by [Ked04b, Prop. 4.1], then $\psi|_{M^{\dagger}}$ induces an injective morphism
$$\psi' : \widetilde{M}^{\dagger} := M^{\dagger} \otimes_{\mathcal{E}^{\dagger}} \widetilde{\mathcal{E}}^{\dagger} \to N \otimes_{\mathcal{E}^{\dagger}} \widetilde{\mathcal{E}}^{\dagger} \to N \otimes_{\mathcal{E}} \widetilde{\mathcal{E}}.$$
The restriction of ψ' to $S^{\mathrm{rev}}_1(M^{\dagger}_{\mathrm{alg}})$ induces a non-trivial morphism
$$S^{\mathrm{rev}}_1(M^{\dagger}_{\mathrm{alg}}) \otimes_{\widetilde{\mathcal{E}}^{\dagger}} \widetilde{\mathcal{E}} \to N \otimes_{\mathcal{E}} \widetilde{\mathcal{E}}.$$
This implies that the slope of $S^{\mathrm{rev}}_1(M^{\dagger}_{\mathrm{alg}})$, which is the maximal slope of M, is s/r. Moreover, by Theorem 4.3.9.(ii), after possibly enlarging r, the dimension of the $\mathbb{Q}_p(p^{1/r})$-vector space $(S^{\mathrm{rev}}_1(M^{\dagger}_{\mathrm{alg}})[p^{1/r}])^{\varphi=p^{s/r}}$ is equal to the rank of $S_m(M)/S_{m-1}(M)$. Similarly, by the Dieudonné–Manin decomposition, $(N \otimes_{\mathcal{E}} \widetilde{\mathcal{E}}[p^{1/r}])^{\varphi=p^{s/r}}$ is a $\mathbb{Q}_p(p^{1/r})$-vector space of dimension equal to the rank of N. We then obtain the inequality of ranks thanks to the injectivity of ψ'. □

4.3.3. *Proof of Proposition 4.3.5.* By the previous reductions, it is enough to prove the result when N is isoclinic of slope s/r and $\overline{N} = M$. In that case, we have to check that $\overline{N}$ has minimal slope s/r and that the inclusion $N \subseteq S_1(\overline{N})$ is an equality. By Lemma 4.3.7, the morphism $\overline{N}^{\vee} \to N^{\vee}$ satisfies the assumptions of Lemma 4.3.10, thus $\overline{N}$ has minimal slope s/r. Moreover, since
$$\mathrm{rk}(N) = \mathrm{rk}(N^{\vee}) \geq \mathrm{rk}(S_m(\overline{N}^{\vee})/S_{m-1}(\overline{N}^{\vee})) = \mathrm{rk}(S_1(\overline{N})),$$
we get $N = S_1(\overline{N})$. □

In order to relate Proposition 4.3.5 with $\mathrm{MS}(\mathbb{A}^1_k)$ we had to prove the following proposition.

Proposition 4.3.11 ([D'Ad23, Prop. 4.2.12]). *Let $N \subseteq M$ be an inclusion of (φ, ∇)-module over $K\langle u\rangle$, where M is $\dagger$-extendable and has constant slopes. The (φ, ∇)-module $\overline{N} \otimes \mathcal{E}$ is the $\dagger$-hull of $N \otimes \mathcal{E}$ in $M \otimes \mathcal{E}$.*

The main tool for this, is an algebraic description of the ring $\mathcal{E}^\dagger$. This viewpoint led to the definition of edged crystalline cohomology in [D'Ad24b]. Let us recall it.

Construction 4.3.12. Let A_n be the image of the morphism $W\langle u_1, \dots, u_n\rangle \to W\langle u\rangle$ which sends $u_i \mapsto \bar{u}_i := pu^i$ and let $\mathfrak{m}_n \subseteq A_n[t]$ be the maximal ideal $(p, t, \bar{u}_1, \dots, \bar{u}_n)$. We denote by $B_n \subseteq \mathcal{O}_{\mathcal{E}}$ the image of $(A_n[t])_{\mathfrak{m}_n} \to \mathcal{O}_{\mathcal{E}}$. The rings A_n and B_n provide integral models of the rings of overconvergent series. More precisely, we have that $\varinjlim_n A_n[\frac{1}{p}] = K\langle u\rangle^\dagger$ and $\varinjlim_n (B_n)^\wedge[\frac{1}{p}] = \mathcal{E}^\dagger$. Note that the A_n-algebra B_n can be also written as

$$\left(\frac{A_n[t]}{(\bar{u}_1 t - p, \dots, \bar{u}_n t^n - p)}\right)_{\mathfrak{m}_n}.$$

Theorem 4.3.13 (See also [Tsu23, Prop. 6.1]). *A smooth curve X over a perfect field k satisfies* $\mathrm{MS}(X)$.

Proof. By the above reductions, we may assume $X = \mathbb{A}^1_k$. Let $N \subseteq M$ be an inclusion of (φ, ∇)-module over $K\langle u\rangle$, where M is $\dagger$-extendable and has constant slopes. By Proposition 4.3.11, we have that $\overline{N} \otimes \mathcal{E}$ is the $\dagger$-hull of $N \otimes \mathcal{E}$ in $M \otimes \mathcal{E}$. Therefore, by Proposition 4.3.5,

$$S_1(N) \otimes \mathcal{E} = S_1(N \otimes \mathcal{E}) = S_1(\overline{N} \otimes \mathcal{E}) = S_1(\overline{N}) \otimes \mathcal{E}.$$

This implies that $S_1(N)$ and $S_1(\overline{N})$ have the same slope and the same rank, so that $S_1(N) = S_1(\overline{N})$. This concludes the proof. □

4.4. **Chevalley theorem and filtrations.** In this section Hypothesis 4.2.1 is in force. Let $\eta \in X(\Omega)$ be a perfect point of X and let $(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}})$ be an overconvergent F^n-isocrystal with constant slopes. Consider the Dieudonné–Manin fibre functor $\omega_{\eta, \mathbb{Q}_p^{\mathrm{ur}}} : \langle \mathcal{M}, \Phi^\infty_{\mathcal{M}}\rangle \to \mathbf{Vec}_{\mathbb{Q}_p^{\mathrm{ur}}}$ associated to η. Write G for $G(\mathcal{M}^\dagger, \Phi^{\dagger,\infty}_{\mathcal{M}}, \eta)$ and H for $G(\mathcal{M}, \Phi^\infty_{\mathcal{M}}, \eta)$.

Definition 4.4.1. For every e, let $\mathbb{G}_m^{1/e}$ be the torus with character group $\frac{1}{e}\mathbb{Z}$ and $\mathbb{G}_m^{1/\infty} := \varprojlim_e \mathbb{G}_m^{1/e}$. If r is the lcm of the denominators of the slopes of $\mathcal{M}$, for every $(\mathcal{M}', \Phi^\infty_{\mathcal{M}'}) \in \langle \mathcal{M}, \Phi^\infty_{\mathcal{M}}\rangle$ we denote by $\tilde{S}_{s/r}(\omega_{\eta,\mathbb{Q}_p^{\mathrm{ur}}}(\mathcal{M}'_\eta, \Phi^\infty_{\mathcal{M}'})) \subseteq \omega_{\eta,\mathbb{Q}_p^{\mathrm{ur}}}(\mathcal{M}'_\eta, \Phi^\infty_{\mathcal{M}'})$ the $\mathbb{Q}_p^{\mathrm{ur}}$-linear vector subspace of slope at most s/r. This defines an exact $\otimes$-filtration $\tilde{S}_\bullet$ of $\omega_{\eta,\mathbb{Q}_p^{\mathrm{ur}}}$ indexed by $\frac{1}{r}\mathbb{Z}$, that in turn defines a morphism $\lambda : \mathbb{G}_m^{1/\infty} \twoheadrightarrow \mathbb{G}_m^{1/r} \to G$ (cf. [Saa72, §2.1.1, page 213]). We say that λ is the *quasi-cocharacter* attached to the slope filtration of $\mathcal{M}_\eta$. We denote by $P_G(\lambda)$ the subgroup of G of those $\otimes$-automorphisms of $\omega_{\eta,\mathbb{Q}_p^{\mathrm{ur}}}$ preserving $\tilde{S}_\bullet$ (as in [*ibid.*, §2.1.3, page 216]).

Proposition 4.4.2. *If* $\mathrm{MS}(X)$ *is true then $H = P_G(\lambda)$. In particular, if G is a reductive group then H is a parabolic subgroup of G.*

Proof. Since $H \subseteq P_G(\lambda)$, we have to prove that $P_G(\lambda) \subseteq H$. By Chevalley's theorem, there exists an overconvergent F^∞-isocrystal $(\mathcal{N}^\dagger, \Phi^{\dagger,\infty}_{\mathcal{N}}) \in \langle \mathcal{M}^\dagger, \Phi^{\dagger,\infty}_{\mathcal{M}}\rangle$ and a rank 1 sub-F^∞-isocrystal $(\mathcal{L}, \Phi^\infty_{\mathcal{L}}) \subseteq (\mathcal{N}, \Phi^\infty_{\mathcal{N}})$, such that H is the stabiliser of the line

$$L := \omega_{\eta,\mathbb{Q}_p^{\mathrm{ur}}}(\mathcal{L}, \Phi_{\mathcal{L}}) \subseteq \omega_{\eta,\mathbb{Q}_p^{\mathrm{ur}}}(\mathcal{N}, \Phi_{\mathcal{N}}) := V.$$

We have to prove that $P_G(\lambda)$ stabilises L. Let $(\overline{\mathcal{L}}, \Phi_{\overline{\mathcal{L}}})$ be the $\dagger$-hull of $(\mathcal{L}, \Phi_{\mathcal{L}}) \subseteq (\mathcal{N}, \Phi_{\mathcal{N}})$ and write $\overline{L}$ for $\omega_{\eta,\mathbb{Q}_p^{\mathrm{ur}}}(\overline{\mathcal{L}}, \Phi_{\overline{\mathcal{L}}})$. We denote by s the slope of $(\mathcal{L}, \Phi_{\mathcal{L}})$ and by $V^{\leq s} \subseteq V$ the subspace of slope smaller or equal than s. Since $\mathrm{MS}(X)$ is satisfied, we know that $\mathcal{L} = S_1(\mathcal{L}) = S_1(\overline{\mathcal{L}})$, which implies that $L = \overline{L} \cap V^{\leq s}$. Since $\overline{\mathcal{L}} \subseteq \mathcal{N}$ admits by definition a $\dagger$-extension, $P_G(\lambda)$ stabilises $\overline{L}$. On the other hand, $P_G(\lambda)$ stabilises $V^{\leq s}$ because $(\mathcal{N}^\dagger, \Phi_{\mathcal{N}}^{\dagger,\infty})$ is an element in $\langle \mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger,\infty} \rangle$. This implies that $P_G(\lambda)$ stabilises L, thus $P_G(\lambda) \subseteq H$ as we wanted. If G is reductive, $H = P_G(\lambda)$ is parabolic by [Saa72, Prop. 2.2.5, page 223]. □

Using Proposition 4.4.2 we deduced the following technical result.

Proposition 4.4.3 ([D'Ad23, Prop. 4.3.5])**.** *Let X be a smooth geometrically connected variety over k such that every dense open $U \subseteq X$ satisfies* $\mathrm{MS}(U)$ *and let $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ be an overconvergent F^n-isocrystal over X. If $(\mathcal{L}, \Phi_{\mathcal{L}}^\infty) \in \langle \mathcal{M}, \Phi_{\mathcal{M}}^\infty \rangle$ is a $\dagger$-extendable rank 1 F^∞-isocrystal, then $(\mathcal{L}^\dagger, \Phi_{\mathcal{L}}^{\dagger,\infty})$ is in $\langle \mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger,\infty} \rangle$.*

In turn, this gives a way to determine $G(\mathcal{M}, \eta)$ knowing $G(\mathcal{M}^\dagger, \eta)$ and $G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta)$.

Corollary 4.4.4. *Under the assumptions of Proposition 4.4.3, the group $G(\mathcal{M}, \eta)$ is equal to the intersection $G(\mathcal{M}^\dagger, \eta) \cap G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta) \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} K$.*

Proof. Since constant F^∞-isocrystals are $\dagger$-extendable, Proposition 4.4.3 implies that

$$G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta)^{\mathrm{cst}} = G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger,\infty}, \eta)^{\mathrm{cst}}.$$

We deduce that the left square of the diagram in Proposition 4.2.9 is cartesian. To get the final result we extend the scalars of the cartesian square from $\mathbb{Q}_p^{\mathrm{ur}}$ to K. Indeed, thanks to Proposition 4.2.11, we know that $G(\mathcal{M}, V_{\mathcal{M}}, \eta) \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} K = G(\mathcal{M}, \eta)$ and $G(\mathcal{M}^\dagger, V_{\mathcal{M}}, \eta) \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} K = G(\mathcal{M}^\dagger, \eta)$. □

4.5. **A Lefschetz theorem.** The main issue to reduce Theorem 4.1.4 to the case of curves is due to the existence of wild ramification in positive characteristic. One would like to find a smooth connected curve $C \subseteq X$ such that for every overconvergent isocrystal $\mathcal{M}^\dagger$ over X, the Tannakian category $\langle \mathcal{M}^\dagger \rangle$ spanned by $\mathcal{M}^\dagger$ is equivalent to the Tannakian category $\langle \mathcal{M}^\dagger|_C \rangle$ spanned by the restriction of $\mathcal{M}^\dagger$ to C. This is possible, for example, for local systems in characteristic 0, or for tamely ramified ℓ-adic lisse sheaves in positive characteristic (see [Esn17]). The failure of the existence of such a nice curve for general ℓ-adic lisse sheaves is already clear for $\mathbb{A}_k^2$ (see [*ibid.*, Lem. 5.4]). On the other hand, if rather than considering all the objects at the same time one focuses on one object at a time, then such a nice curve exists over finite fields both for ℓ-adic lisse sheaves and overconvergent F^n-isocrystals (see [Kat99, Lem. 6 and Thm. 8] and [AE19, Thm. 3.10]). We extended this result to *docile* overconvergent F^n-isocrystals over general perfect fields, namely those overconvergent F^n-isocrystals which admit a log-extension with nilpotent residues.

Theorem 4.5.1 ([D'Ad23, Thm. 4.4.3])**.** *Let $Y \subseteq \mathbb{P}_k^d$ be a smooth connected projective variety of dimension at least 2 and let $D \subseteq Y$ be a simple normal crossing divisor. If $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ is an overconvergent F^n-isocrystal over $X := Y \setminus D$ docile along D, then there exists a smooth connected curve $C \subseteq X$ such that the restriction functor $\langle \mathcal{M}^\dagger \rangle \to \langle \mathcal{M}^\dagger|_C \rangle$ is an equivalence of categories.*

This theorem is obtained by a combination of various Lefschetz-type results. One of the main ingredients is [AE19, Cor. 2.4], proven by Abe–Esnault, which gives a class of curves C such that the restriction functor $\langle \mathcal{M}^\dagger \rangle \to \langle \mathcal{M}^\dagger|_C \rangle$ is fully faithful. To prove Theorem 4.5.1, we show that for at least one of these curves the restriction functor is also essentially surjective. This condition can be tested on rank 1 objects, which have the advantage of coming from p-adic characters of the étale fundamental group and they are easier to extend from C to X. The difficult part is to impose that the extended characters come from overconvergent F^n-isocrystals. In our proof, this is done by combining Proposition 4.4.3 and the following lemma on the ramification of rank 1 lisse $\overline{\mathbb{Q}_p}$-sheaves.

Lemma 4.5.2. *Let X be a smooth connected variety over k, $D \subseteq X$ an irreducible divisor and $\mathcal{V}$ a lisse $\overline{\mathbb{Q}}_p$-sheaf over $X \setminus D$. There exists a dense smooth open $D' \subseteq D$ and a conic closed subscheme $Z \subseteq TX \times_X D'$ of codimension 1 at every fibre which satisfies the following property.*

> $\mathrm{R}_{\mathcal{V}}(Z)$: *Let $C \subseteq X$ be a smooth curve not contained in D and intersecting D' at some closed point x such that TC_x is not contained in Z_x. For every rank 1 lisse sheaf $\mathcal{L} \in \langle \mathcal{V} \rangle$ ramified at D, $\mathcal{L}|_C$ is ramified at x.*

Interestingly, in order to use Proposition 4.4.3 we need Theorem 4.1.4 for curves. Therefore, the proofs of Theorem 4.1.4 and Theorem 4.5.1 are intrinsically intertwined.

5. The local enhancement of the conjecture

5.1. **Statement and strategy of the proof.** Let X be a smooth irreducible variety over a perfect field with a closed point x and let $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}^\dagger})$ be a semi-simple overconvergent F-isocrystal over X with constant Newton polygon. Since the Newton polygon is constant, the associated F-isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ admits the slope filtration. Thanks to the parabolicity conjecture, we know that if $\mathcal{M}^\dagger$ is a semi-simple overconvergent F-isocrystal, then $G(\mathcal{M}, x) \subseteq G(\mathcal{M}^\dagger, x)$ is a parabolic subgroup $P \subseteq G(\mathcal{M}^\dagger, x)$. By the Levi decomposition, we can write P as the semi-direct product $U \rtimes L$ with U the unipotent radical and L a Levi subgroup of P. The group L is isomorphic to the monodromy group of the graded object associated to $(\mathcal{M}, \Phi_{\mathcal{M}})$. By [BM90, Thm. 2.4.1] and the invariance of the étale site with respect to the perfection morphism $X^{\mathrm{perf}} \to X$, the algebraic group L is also isomorphic to the monodromy group of $\mathcal{M}|_{X^{\mathrm{perf}}}$. In this section we want to explain how U can be obtain as well as the monodromy group of a certain base change of $\mathcal{M}$.

Let $X^{/x}$ be the formal completion of X at x and write $G(\mathcal{M}^{/x})$ for the monodromy group of the restriction of $\mathcal{M}$ to $X^{/x}$.

Theorem 5.1.1 ([DvH22, Thm. 3.4.4])**.** *The monodromy group $G(\mathcal{M}^{/x}, x)$ of the restriction of $\mathcal{M}$ to $X^{/x}$ is the unipotent radical of the monodromy group $G(\mathcal{M}, x)$.*

When $\mathcal{M}$ is the crystalline Dieudonné-module of an ordinary p-divisible group this result is proved by Chai by doing explicit computations with Serre–Tate coordinates. Our proof builds instead on the techniques developed in [D'Ad23] and uses new descent results for isocrystals from [Dri22],[Mat22].

Since $X^{/x}$ is geometrically simply connected, each isocrystal underlying an isoclinic F-isocrystal over $X^{/x}$ is trivial by [BM90, Thm. 2.4.1]. This already implies that $G(\mathcal{M}^{/x}, x)$ is unipotent. To relate $G(\mathcal{M}^{/x}, x)$ and the unipotent radical of $G(\mathcal{M}, x)$, we pass through the respective generic points.

Write k for the function field of X and k_x for the function field of $X^{/x}$. We first prove in Theorem 5.2.1 that passing from X to $\mathrm{Spec}(k)$ we do not change the monodromy group of $\mathcal{M}$, as in the étale setting. Then we show that if we extend k to k^{sep} the monodromy group of $\mathcal{M}$ becomes the unipotent radical of $G(\mathcal{M}, x)$ (Proposition 5.4.2). This means that the extension of scalars from k to k^{sep} kills precisely a Levi subgroup of $G(\mathcal{M}, x)$. Subsequently, using the fact that the field extension $k \subseteq k_x$ is separable, we show that when we extend F-isocrystals from k^{sep} to k_x^{sep}, their slope filtration does not acquire new splittings (Proposition 5.3.5). This is enough to prove that the local monodromy group $G(\mathcal{M}^{/x}, x)$ is the same as the monodromy group over k^{sep}. By the previous part of the argument we then deduce that $G(\mathcal{M}^{/x}, x)$ is precisely the unipotent radical of $G(\mathcal{M}, x)$.

5.2. **Monodromy group at the generic point.** As we mentioned, in the previous section, we proved Theorem 5.1.1 passing through the generic point $\eta \in X$. For this purpose we proved the following result of independent interest.

Theorem 5.2.1 ([DvH22, Thm. 3.2.8])**.** *Let X be an irreducible Noetherian Frobenius-smooth scheme over $\mathbb{F}_p$. If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is an F-isocrystal over X with slope filtration, then $G(\mathcal{M}, \eta^{\mathrm{perf}}) = G(\mathcal{M}_\eta, \eta^{\mathrm{perf}})$.*

Proof. Let K be the fraction field of $W(\kappa)$ with κ the field of constants of X and let K' be the fraction field of the ring of Witt vectors of η^{perf}. Thanks to [Sta08, Prop. 3.1.8] applied with $F = K, F' = K$, and $F'' = K'$, it is enough to prove that $\langle \mathcal{M} \rangle \to \langle \mathcal{M}_\eta \rangle$ is fully faithful and sends semi-simple objects to semi-simple objects. The first part is proven in Theorem 3.3.4 and does not need the assumption on the slope filtration. For the second part we have to prove that for every irreducible $\mathcal{N} \in \langle \mathcal{M} \rangle$, the base change $\mathcal{N}_\eta$ is semi-simple.

Since $\mathcal{N}$ is irreducible, it is a subquotient of $\mathcal{M}^{\otimes m} \otimes (\mathcal{M}^\vee)^{\otimes n}$ for some $m, n \geq 0$ and by the assumption $\mathcal{M}^{\otimes m} \otimes (\mathcal{M}^\vee)^{\otimes n}$ can be endowed with a Frobenius structure with slope filtration. After taking the sth-power of the Frobenius structure for some $s > 0$ and making a Tate twist, we may further assume that $\mathcal{N}$ appears in the unit-root part of an F^s-isocrystal. Therefore, taking a Jordan–Hölder filtration, we may assume that $\mathcal{N}$ is a subquotient of an isocrystal $\mathcal{M}'$ which admits a unit-root F^s-structure $\Phi_{\mathcal{M}'}$ such that $(\mathcal{M}', \Phi_{\mathcal{M}'})$ is semi-simple. By [BM90, Thm. 2.4.1], the F^s-isocrystal $(\mathcal{M}', \Phi_{\mathcal{M}'})$ corresponds to a semi-simple lisse $\mathbb{Q}_{p^s}$-sheaf over X. By the regularity of X, the lisse sheaf remains semi-simple when restricted to the generic point. This implies that $(\mathcal{M}'_\eta, \Phi_{\mathcal{M}'_\eta})$ is semi-simple. To conclude we have to prove that $\mathcal{M}'_\eta$ is semi-simple as well. Let $\mathcal{N}'_\eta \subseteq \mathcal{M}'_\eta$ be the *socle* of $\mathcal{M}'_\eta$, namely the sum of all the irreducible subobjects of $\mathcal{M}'_\eta$. By maximality, $\mathcal{N}'_\eta$ is stabilised by the F^s-structure, thus it upgrades to a subobject $(\mathcal{N}'_\eta, \Phi_{\mathcal{N}'_\eta}) \subseteq (\mathcal{M}'_\eta, \Phi_{\mathcal{M}'_\eta})$. By semi-simplicity, the inclusion admits a retraction, which induces in particular a retraction of $\mathcal{N}'_\eta \subseteq \mathcal{M}'_\eta$. This implies that $\mathcal{N}'_\eta = \mathcal{M}'_\eta$, as we wanted. □

5.3. **Descent for isocrystals.** We see now various descent results that we will need in the next section for (F-)isocrystals. Let $f : Y \to X$ be a pro-étale Π-cover of Noetherian Frobenius-smooth schemes over $\mathbb{F}_p$ where Π is a profinite group and let $y \in Y(\Omega)$ be an Ω-point of Y with Ω a perfect field.

Lemma 5.3.1. *For every isocrystal $\mathcal{M}$ over X, the maximal trivial subobject of $f^*\mathcal{M}$ descends to a subobject $\mathcal{N} \subseteq \mathcal{M}$. Moreover, if $\mathcal{M}$ is endowed with a Frobenius structure $\Phi_{\mathcal{M}}$, the inclusion $\mathcal{N} \subseteq \mathcal{M}$ upgrades to an inclusion $(\mathcal{N}, \Phi_{\mathcal{N}}) \subseteq (\mathcal{M}, \Phi_{\mathcal{M}})$ of F-isocrystals and $(\mathcal{N}, \Phi_{\mathcal{N}})$ is a direct sum of isoclinic F-isocrystals.*

Proof. Since the cover $Y \to X$ is a quasi-syntomic cover, it satisfies descent for isocrystals thanks to [Dri22, Prop. 3.5.4] (see also [Mat22] or [BS23, §2]). By the assumption,

$$Y \times_X Y \simeq \varprojlim_{U \subseteq \Pi} (Y \times_X Y)^U$$

where the limit runs over all the open normal subgroups of Π and $(Y \times_X Y)^U := \coprod_{[\gamma] \in \Pi/U} Y_{[\gamma]}$ is a disjoint union of copies of Y. The group Π acts on $Y \times_X Y$ in the obvious way. Since $f^*\mathcal{M}$ comes from X, it is endowed with a descent datum with respect to the cover $Y \to X$. This datum consists of isomorphisms $\gamma^*\mathcal{M}_{(Y \times_X Y)^U} \simeq \mathcal{M}_{(Y \times_X Y)^U}$ for each $U \subseteq \Pi$ and $\gamma \in \Pi$. The functor γ^* sends trivial objects to trivial objects, which implies that the descent datum restricts to a descent datum of $\mathcal{T}$, the maximal trivial subobject of $f^*\mathcal{M}$. Therefore, $\mathcal{T}$ descends to a subobject $\mathcal{N} \subseteq \mathcal{M}$, as we wanted. If $\mathcal{M}$ is endowed with a Frobenius structure, then it induces a Frobenius structure on each isocrystal $\mathcal{M}_{(Y \times_X Y)^U}$ and this structure preserves each maximal trivial subobject of given slope. This implies that the descended object $\mathcal{N} \subseteq \mathcal{M}$ is stabilised as well by the Frobenius and the induced Frobenius structure satisfies the desired property. $\square$

Proposition 5.3.2. *Let $(\mathcal{M}, \Phi_{\mathcal{M}})$ be an F-isocrystal with the slope filtration and write ν for the associated Newton cocharacter. If $R_u(G(\mathcal{M}, f(y))) \subseteq U_\nu$ and $\mathrm{Gr}_{S_\bullet}(f^*\mathcal{M})$ is trivial, then $G(f^*\mathcal{M}, y) = R_u(G(\mathcal{M}, f(y)))$.*

Proof. Since $\mathrm{Gr}_{S_\bullet}(f^*\mathcal{M})$ is trivial, the group $G(f^*\mathcal{M}, y)$ is a unipotent subgroup of $G(\mathcal{M}, f(y))$ sitting inside U_ν. Therefore, we are in the situation of [D'Ad23, Prop. 3.1.2] and we have to prove that (ii) is satisfied. This amounts to show that for every $m, n \geq 0$, the maximal trivial subobject $\mathcal{T} \subseteq f^*(\mathcal{M}^{\otimes m} \otimes (\mathcal{M}^\vee)^{\otimes n})$ descends to a semi-simple isocrystal over X. By Lemma 5.3.1, we know that $\mathcal{T}$ descends to an isocrystal $\mathcal{N}$ which is the direct sum of isocrystals which can be endowed with an isoclinic Frobenius structure. Since $R_u(G(\mathcal{M}, f(y)))$ is contained in U_ν, we deduce that $\mathcal{N}$ is semi-simple, as we wanted. $\square$

Lemma 5.3.3. *If k'/k is a separable field extension and k' admits a finite p-basis, then $k' \otimes_k k'$ admits a finite p-basis as well.*

Proof. Thanks to [Mat70, Thm. 26.6], the field k admits a finite p-basis $t_1, \cdots, t_d$ which extends to a finite p-basis $t_1, \cdots, t_d, u_1, \cdots u_e$ of k'. We claim that $\Gamma := \{t_i \otimes 1\}_i \cup \{u_i \otimes 1\}_i \cup \{1 \otimes u_i\}_i$ is a finite p-basis of $k' \otimes_k k'$. It is clear from the construction that the elements of Γ generate $k' \otimes_k k'$ over $(k' \otimes_k k')^p$. On the other hand, the exact sequence

$$0 \to \Omega^1_{k/\mathbb{F}_p} \otimes_k (k' \otimes_k k') \to \Omega^1_{k' \otimes_k k'/\mathbb{F}_p} \to (\Omega^1_{k'/k} \otimes_k k') \oplus (k' \otimes_k \Omega^1_{k'/k}) \to 0$$

shows that the elements $d\gamma$ with $\gamma \in \Gamma$ form a basis of the free module $\Omega^1_{k' \otimes_k k'/\mathbb{F}_p}$. We deduce the p-independence of the elements of Γ by arguing as in [Stacks, Tag 07P2]. $\square$

Lemma 5.3.4. *Let X be a Frobenius-smooth scheme over $\mathbb{F}_p$. For every F-isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ with a locally-free lattice and constant Newton polygon such that all the slopes are different from 0, the vector space of global sections fixed by the Frobenius structure is trivial.*

Proof. Since X is Frobenius-smooth, by [BM90, §1.3.5.ii] the global sections of any isocrystal over X embed into the global sections of the base change to X^{perf}. Over X^{perf} we argue as in the proof of [Kat79, Thm. 2.4.2], namely we assume $X^{\mathrm{perf}} = \mathrm{Spec}(A)$ affine and we embed A into a product of perfect fields. This reduces the problem to the case of perfect fields, where the result is well-known. □

Proposition 5.3.5. *Let $k \subseteq k'$ a separable extension of characteristic p fields with finite p-basis and let $(\mathcal{M}, \Phi_{\mathcal{M}})$ be a free F-isocrystal over k with slope filtration $S_\bullet$ of length n. If $\mathcal{M}_{k'}$ admits a Frobenius-stable splitting of the form $\mathcal{N}_{k'} \oplus S_{n-1}(\mathcal{M}_{k'})$ with $\mathcal{N}_{k'}$ some subobject of $\mathcal{M}_{k'}$, the same is true for $\mathcal{M}$.*

Proof. Since $\mathrm{Spec}k' \to \mathrm{Spec}k$ is a quasi-syntomic cover, it satisfies descent for isocrystals thanks to the descent results of Drinfeld and Mathew in [Dri22], [Mat22] (see [BS23, Thm. 2.2]). Therefore, in order to descend $\mathcal{N}_{k'}$ to k it is enough to show that the splitting $\mathcal{N}_{k'\otimes_k k'} \oplus S_{n-1}(\mathcal{M}_{k'\otimes_k k'})$ is unique. Suppose that $\mathcal{N}'_{k'\otimes_k k'} \oplus S_{n-1}(\mathcal{M}_{k'\otimes_k k'})$ was a different splitting. Then there would exist a non-trivial Frobenius-equivariant morphism $\mathcal{N}'_{k'\otimes_k k'} \to S_{n-1}(\mathcal{M}_{k'\otimes_k k'})$. In other words, the F-isocrystal $\underline{\mathrm{Hom}}(\mathcal{N}'_{k'\otimes_k k'}, S_{n-1}(\mathcal{M}_{k'\otimes_k k'}))$ would have a non-trivial Frobenius-invariant global section. Since the slopes of $\underline{\mathrm{Hom}}(\mathcal{N}'_{k'\otimes_k k'}, S_{n-1}(\mathcal{M}_{k'\otimes_k k'}))$ are all negative by definition and $k' \otimes_k k'$ admits a finite p-basis by Lemma 5.3.3, this would contradict Lemma 5.3.4. □

5.4. **Proof of the local enhancement.** We are ready to put all the previous results together and prove Theorem 5.1.1. Let X be a smooth irreducible variety over a perfect field and let x be a closed point of X. We denote by k the function field of X and by k_x the function field of $X^{/x}$. We also write η^{sep} (resp. $\bar{\eta}$) for the points over the generic point of X associated to a separable (resp. algebraic) closure of k.

Lemma 5.4.1. *The fields k and k_x have a common finite p-basis. In particular, $k \subseteq k_x$ is a separable field extension.*

Proof. By [Mat70, Thm. 26.7], it is enough to show that $\Omega^1_{k/\mathbb{F}_p} \otimes_k k_x = \Omega^1_{k_x/\mathbb{F}_p}$. Write A for the local ring of X at x and $A^\wedge_x$ for the completion with respect to the maximal ideal $\mathfrak{m}_x$. Since A is regular, thanks to [Mat70, Thm. 30.5 and Thm. 30.9], we deduce that $\Omega^1_{A/\mathbb{F}_p} \otimes_A A^\wedge_x = \Omega^1_{A^\wedge_x/\mathbb{F}_p}$. We get the desired result after inverting $\mathfrak{m}_x - \{0\}$. □

Proposition 5.4.2. *If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is an F-isocrystal over X such that $R_u(G(\mathcal{M}, \bar{\eta})) \subseteq U_\nu$, then $G(\mathcal{M}_{\eta^{\mathrm{sep}}}, \bar{\eta}) = R_u(G(\mathcal{M}, \bar{\eta}))$.*

Proof. By Theorem 5.2.1 we have that $G(\mathcal{M}, \bar{\eta}) = G(\mathcal{M}_\eta, \bar{\eta})$, so that we are reduced to prove the statement for $G(\mathcal{M}_\eta, \bar{\eta})$. Note that the cover $f : \eta^{\mathrm{sep}} \to \eta$ is a pro-étale $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-cover and $\mathrm{Gr}_{S_\bullet}(f^*\mathcal{M}_\eta)$ is trivial because η^{sep} is simply connected. This shows that we can apply Proposition 5.3.2 and deduce the desired result. □

Proposition 5.4.3. *If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is an F-isocrystal over X coming from an irreducible overconvergent F-isocrystal with constant Newton polygon, then $H^0(X^{/x}, (S_1(\mathcal{M}))^{/x}) = H^0(X^{/x}, \mathcal{M}^{/x})$.*

Proof. By Galois descent we may assume that the ring of constants of X is an algebraically closed field. The inclusion $H^0(X^{/x},(S_1(\mathcal{M}))^{/x}) \subseteq H^0(X^{/x},\mathcal{M}^{/x})$ is an inclusion of F-isocrystals over κ. We suppose by contradiction that this is not an equality. Let $q_r > q_1$ be the greatest slope appearing in $H^0(X^{/x},\mathcal{M}^{/x})$ and let v be a non-zero vector such that $\Phi^n_{\mathcal{M}^{/x}}(v) = p^{q_r n}v$ for $n \gg 0$. Write $(\tilde{\mathcal{M}},\Phi_{\tilde{\mathcal{M}}})$ for the base change of $(\mathcal{M},\Phi_{\mathcal{M}})$ to η^{sep}.

By the parabolicity conjecture $R_u(G(\mathcal{M},\bar{\eta}))$ is contained in U_ν because $(\mathcal{M},\Phi_{\mathcal{M}})$ comes from an irreducible overconvergent F-isocrystal. Proposition 5.4.2 then implies that the monodromy group $G(\tilde{\mathcal{M}},\bar{\eta})$ is equal to $G(\mathcal{M},\bar{\eta})\cap U_\nu$. Therefore, the line spanned by v determines a rank 1 subobject $\tilde{\mathcal{L}} \subseteq S_r(\tilde{\mathcal{M}})/S_{r-1}(\tilde{\mathcal{M}})$ stabilised by the Frobenius. The preimage of this isocrystal in $S_r(\tilde{\mathcal{M}})$, denoted by $\tilde{\mathcal{N}}$, is kept invariant by the Frobenius and sits in an exact sequence

$$0 \to S_{r-1}(\tilde{\mathcal{M}}) \to \tilde{\mathcal{N}} \to \tilde{\mathcal{L}} \to 0.$$

Since $(\mathcal{M},\Phi_{\mathcal{M}})$ comes from an irreducible overconvergent F-isocrystal, the sequence does not admit a Frobenius-equivariant splitting by [D'Ad23, Thm. 4.1.3]. By Proposition 5.3.5, the base change of this extension to k_x^{sep} does not split as well. This leads to a contradiction since v is a vector in $H^0(X^{/x},\mathcal{M}^{/x})$ which produces a non-trivial global section of $\tilde{\mathcal{N}} \subseteq \tilde{\mathcal{M}}$. □

We write η_x for the generic point of $X^{/x}$ and $G(\mathcal{M}^{/x},\eta_x^{\mathrm{perf}})$ for the monodromy group of $\mathcal{M}^{/x}$ with respect to the perfection of η_x. We can finally prove the following theorem.

Theorem 5.4.4 ([DvH22, Thm. 3.4.4])**.** *If $(\mathcal{M},\Phi_{\mathcal{M}})$ comes from a semi-simple overconvergent isocrystal endowed with a Frobenius structure with constant Newton polygon, then*

$$G(\mathcal{M}^{/x},\eta_x^{\mathrm{perf}}) = R_u(G(\mathcal{M},\eta_x^{\mathrm{perf}})).$$

Proof. Write G for the group $G(\mathcal{M},\eta)$ and V for the induced G-representation. By the parabolicity conjecture we have that $R_u(G)$ is contained in U_ν where ν is the Newton cocharacter. Since $X^{/x}$ is geometrically simply connected we deduce that $\mathrm{Gr}_{S_\bullet}(\mathcal{M})^{/x}$ is trivial. This implies that $G(\mathcal{M}^{/x},\eta_x^{\mathrm{perf}}) \subseteq R_u(G) \subseteq U_\nu$. Therefore, in order to apply the criterion of [D'Ad23, Prop. 3.1.2] it is enough to show that for every $\mathcal{N} \in \langle\mathcal{M}\rangle$, the space of global sections of $\mathcal{N}^{/x}$ is the same as the fibre at x of some direct sum of isoclinic subobjects of $\mathcal{N}$. To prove this, we may assume that $\mathcal{N}$ can be endowed with a Frobenius structure $\Phi_{\mathcal{N}}$ and $(\mathcal{N},\Phi_{\mathcal{N}})$ is irreducible. Thanks to Proposition 5.4.3, we deduce that the fibre of $S_1(\mathcal{N})$ at x is the same as $H^0(X^{/x},\mathcal{N}^{/x})$. This yields the desired result. □

6. The Hecke orbit conjecture

6.1. **Strategy of the proof.** The overall structure of the proof of Theorem 1.3.2 is similar to the proof of the ordinary Hecke orbit conjecture in [vH24] and is based on a strategy implicit in the work of Chai–Oort and sketched to us by Chai in a letter.

To explain the proof we first need to establish some notation. Let $Z \subseteq C$ be a reduced closed subvariety that is stable under the prime-to-p Hecke operators, as in the statement of Theorem 1.3.2. The central leaf is contained in a Newton stratum $\mathrm{Sh}_{\mathsf{G},U,[b]} \subseteq \mathrm{Sh}_{\mathsf{G},U}$ which has an associated Newton fractional cocharacter

$$\nu_b \colon \mathbb{G}_m^{1/\infty} \to G.$$

Attached to this cocharacter is a parabolic subgroup P_{ν_b} with unipotent radical U_{ν_b}.

Let Z^{sm} be the smooth locus of Z. Then Corollary 3.3.3 of [vH24] tells us that the monodromy group of the crystalline Dieudonné module $\mathcal{M}$ of the universal p-divisible group over Z^{sm} is isomorphic to P_{ν_b}. Theorem 5.1.1 then tells us that for $x \in Z^{\mathrm{sm}}(\overline{\mathbb{F}}_p)$, the monodromy group of $\mathcal{M}$ both over $Z^{/x}$ and over $C^{/x}$ is equal to U_{ν_b}. We want to show that $Z^{/x} = C^{/x}$, which allows us to conclude that Z^{sm} and hence Z are equidimensional of the same dimension as C.

For this we construct generalised Serre–Tate coordinates on the formal completion $C^{/x}$. To be precise, we show that there is a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra[9] $\mathfrak{a}^+$ over $\breve{\mathbb{Z}}_p$ governing the structure of $C^{/x}$. For example, if $\mathrm{Sh}_{\mathsf{G},U}$ is a Siegel modular variety and C is the ordinary locus, then $C^{/x}$ is a p-divisible formal group by the classical theory of Serre–Tate coordinates, and $\mathfrak{a}^+ = \mathbb{D}(C^{/x})$ is its Dieudonné module, equipped with the trivial Lie bracket.

We note that $C^{/x}$ is a formal homogeneous for a formal group $\tilde{\Pi}(\mathfrak{a})$ attached to the nilpotent Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra $\mathfrak{a} := \mathfrak{a}^+[\frac{1}{p}]$. The stabiliser is a profinite group scheme $\Pi(\mathfrak{a}^+)$ over $\overline{\mathbb{F}}_p$. The formal scheme underlying $\tilde{\Pi}(\mathfrak{a})$ is the universal cover of the p-divisible group associated to $\mathfrak{a}^+$. In the Siegel case, this unipotent formal group is the identity component of the group of self-quasi isogenies of the p-divisible group $A_x[p^\infty]$ which are compatible with the polarisation up to a scalar.

The Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra $\mathfrak{a}$ turns out to be isomorphic to $\mathrm{Lie}(U_{\nu_b})$ equipped with a natural F-structure. For a Dieudonné–Lie $\breve{\mathbb{Q}}_p$-subalgebra $\mathfrak{b} \subseteq \mathfrak{a}$, we constructed a formally smooth closed formal subscheme $Z(\mathfrak{b}^+) \subseteq C^{/x}$. This formal subscheme is again a formal homogeneous space under a unipotent formal group $\tilde{\Pi}(\mathfrak{b}) \subseteq \tilde{\Pi}(\mathfrak{a})$. Chai and Oort proved that $Z^{/x}$ admits such a description.

Theorem 6.1.1 ([CO22])**.** *There is an F-stable Lie subalgebra $\mathfrak{b} \subseteq \mathfrak{a}$ such that $Z^{/x} = Z(\mathfrak{b}^+)$.*

To prove Theorem 1.3.2, we are then reduced to show that the restriction of $\mathcal{M}$ to formal subschemes $Z(\mathfrak{b}^+) \subseteq C^{/x}$ for $\mathfrak{b} \subsetneq \mathfrak{a}$ have smaller monodromy.

Theorem 6.1.2 (Theorem 6.5.1)**.** *The Lie algebra of the monodromy group of $\mathcal{M}$ over $Z(\mathfrak{b}^+)$ is contained in $\mathfrak{b}$.*

By the previous discussion, we know that the Lie algebra of the monodromy group of $\mathcal{M}$ over $Z^{/x}$ is equal to $\mathfrak{a}$. Therefore, if $Z^{/x} = Z(\mathfrak{b}^+)$ for some $\mathfrak{b}$, then $\mathfrak{a} \subseteq \mathfrak{b} \subseteq \mathfrak{a}$, so that $Z^{/x} = Z(\mathfrak{b}^+) = Z(\mathfrak{a}^+) = C^{/x}$.

The proof of Theorem 6.1.2 uses the Cartier–Witt stacks of Drinfeld [Dri20] and Bhatt–Lurie [BL22] in combination with the interpretation of $C^{/x}$ as a formal deformation space of the trivial torsor for the profinite group scheme $\Pi(\mathfrak{a}^+)$, due to Chai–Oort [Cha20] in the PEL case. In particular, we show that the closed formal subscheme $Z(\mathfrak{b}^+) \subseteq C^{/x}$ can be identified with the formal deformation space of the trivial torsor for a certain closed subgroup $\Pi(\mathfrak{b}^+) \subseteq \Pi(\mathfrak{a}^+)$.

Remark 6.1.3. The unipotent formal group $\tilde{\Pi}(\mathfrak{a})$ is closely related to the "unipotent group diamond" $\tilde{G}_b^{>0}$ introduced in Chapter III.5 of Fargues–Scholze, [FS21]. To be precise, there should be an isomorphism $\tilde{\Pi}(\mathfrak{a})^\diamond \simeq \tilde{G}_b^{>0}$ of v-sheaves in groups over $\mathrm{Spd}(\overline{\mathbb{F}}_p)$.

[9] See Definition 6.2.1.

6.2. **Dieudonné–Lie algebras.** One of the key tools we use in the proof is a generalisation of Serre–Tate coordinates. This is done using the new notion of *Dieudonné–Lie algebras*. In §6.3 we will see how to attach to a certain class of Dieudonné–Lie algebras, that we call *plain Dieudonné–Lie algebras*, a formal homogeneous space.

We write $\breve{\mathbb{Z}}_p$ for $W(\overline{\mathbb{F}}_p)$ and $\breve{\mathbb{Q}}_p$ for its fraction field. We also write φ for the Frobenius lift on both $\breve{\mathbb{Z}}_p$ and $\breve{\mathbb{Q}}_p$.

Definition 6.2.1. A *Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra* is a triple $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$ where $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+})$ is a Dieudonné module over $\overline{\mathbb{F}}_p$ (see Definition 3.4.7) and

$$[-,-] : \mathfrak{a}^+ \times \mathfrak{a}^+ \to \mathfrak{a}^+$$

is a Lie bracket such that the following diagram commutes

$$\begin{tikzcd} \mathfrak{a}^+[\frac{1}{p}] \times \mathfrak{a}^+[\frac{1}{p}] \arrow[r, "\varphi_{\mathfrak{a}^+} \times \varphi_{\mathfrak{a}^+}"] \arrow[d, "{[-,-]}"] & \mathfrak{a}^+[\frac{1}{p}] \times \mathfrak{a}^+[\frac{1}{p}] \arrow[d, "{[-,-]}"] \\ \mathfrak{a}^+[\frac{1}{p}] \arrow[r, "\varphi_{\mathfrak{a}^+}"] & \mathfrak{a}^+[\frac{1}{p}]. \end{tikzcd}$$

A morphism of Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras is a $\breve{\mathbb{Z}}_p$-linear map $f : \mathfrak{a}^+ \to \mathfrak{b}^+$ that respects the Lie brackets and induces a homomorphism of Dieudonné modules. If f is injective with finite cokernel we say that f is an *isogeny*. We write $\mathbb{X}(\mathfrak{a}^+)$ for the p-divisible group attached to the Dieudonné module $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+})$. We also say that a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra is *completely slope divisible* if the underlying Dieudonné module is so (see Definition 3.6.1).
Similarly, a *Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra* is a triple $(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$ where $(\mathfrak{a}, \varphi_{\mathfrak{a}})$ is a rational Dieudonné module over $\overline{\mathbb{F}}_p$ and $[-,-]$ is a Lie bracket of $\mathfrak{a}$ satisfying the same compatibility. We write

$$(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$$

for the Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra obtained from a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra

$$(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$$

by inverting p. We also denote by $\tilde{\mathbb{X}}(\mathfrak{a})$ the universal cover of the p-divisible group associated to an integral lattice of $(\mathfrak{a}, \varphi_{\mathfrak{a}})$. This assignment does not depend on the choice of the lattice. We will very often omit the Frobenius structure and the Lie bracket in the notation of Dieudonné–Lie algebras.

Remark 6.2.2. Alternatively, Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras (resp. $\breve{\mathbb{Q}}_p$-algebras) can be defined as the Lie algebra objects in the symmetric monoidal category of F-crystals (resp. F-isocrystals) over $\overline{\mathbb{F}}_p$ such that the underlying F-crystal (resp. F-isocrystals) is a Dieudonné module.

Example 6.2.3. The first example of a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra is the Dieudonné module of the internal-Hom p-divisible group $\mathcal{H}_Y$ attached to a p-divisible group Y over $\overline{\mathbb{F}}_p$, denoted by $\mathbb{D}(\mathcal{H}_Y)$. Indeed, the Lie bracket coming from the commutator bracket on $T_p\mathcal{H}_Y = \mathbf{Hom}(Y,Y)$, induces an φ-equivariant Lie bracket on $\mathfrak{a}$. The Lie bracket on $\mathbf{Hom}(Y,Y)$ clearly sends the identity component $\mathbf{Hom}(Y,Y)^\circ$ to itself. This leads to our second example of a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, the one induced on $\mathbb{D}(\mathcal{H}_Y^\circ)$.

Given a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, there is also a natural procedure to get many others isogenous Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras by using the Frobenius structure. Let us see this more in details, as it will play an important role.

Construction 6.2.4. For a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$ and $n \in \mathbb{Z}$, we define $\Phi^n(\mathfrak{a}^+)$ to be the $\breve{\mathbb{Z}}_p$-submodule $\varphi^n_{\mathfrak{a}^+}(\mathfrak{a}^+) \subseteq \mathfrak{a}$. This $\breve{\mathbb{Z}}_p$-submodule is preserved by the Frobenius and the Lie bracket of $\mathfrak{a}$. We then get on $\Phi^n(\mathfrak{a}^+)$ a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra structure. Note that $\varphi_{\mathfrak{a}}^{-n}$ induces an isomorphism $\Phi^n(\mathfrak{a}^+) \xrightarrow{\sim} \mathfrak{a}^{+,(p^n)}$ of Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras.

Lemma 6.2.5. *For $n \geq 0$, there is a natural exact sequence of fpqc sheaves*

$$0 \to T_p\mathbb{X}(\mathfrak{a}^+) \to T_p\mathbb{X}(\Phi^n(\mathfrak{a}^+)) \to \mathbb{X}(\mathfrak{a}^+)[F^n] \to 0.$$

Proof. Thanks to the isomorphism $\varphi_{\mathfrak{a}}^{-n} \colon \Phi^n(\mathfrak{a}^+) \xrightarrow{\sim} \mathfrak{a}^{+,(p^n)}$, this is equivalent to proving that we have an exact sequence

$$0 \to T_p\mathbb{X}(\mathfrak{a}^+) \to T_p\mathbb{X}(\mathfrak{a}^+)^{(p^n)} \to \mathbb{X}(\mathfrak{a}^+)[F^n] \to 0,$$

where $T_p\mathbb{X}(\mathfrak{a}^+) \to T_p\mathbb{X}(\mathfrak{a}^+)^{(p^n)}$ is induced by the nth-power of the (relative) Frobenius of $\mathbb{X}(\mathfrak{a}^+)$. The result then follows by a classical diagram chasing, using the fact that the Frobenius of $\mathbb{X}(\mathfrak{a}^+)$ is faithfully flat. □

The fundamental lemma we will use very often to study Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebras and reduce ourself to the abelian case is the following one.

Lemma 6.2.6. *Let $\mathfrak{a}$ be a Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra where all the slopes are negative. Let μ_1 be the smallest slope of $\mathfrak{a}$ and let $\mathfrak{b} \subseteq \mathfrak{a}$ be an F-stable $\breve{\mathbb{Q}}_p$-subspace that is isoclinic of that slope. Then $\mathfrak{b}$ is contained in the centre of $\mathfrak{a}$.*

Proof. There are no nonzero F-equivariant maps $\mathfrak{b} \otimes \mathfrak{a} \to \mathfrak{a}$ because, by assumption, all the slopes of $\mathfrak{b} \otimes \mathfrak{a}$ are strictly smaller than the slopes of $\mathfrak{a}$. Hence the restriction of the Lie bracket to $\mathfrak{b} \times \mathfrak{a}$ is trivial. □

6.2.1. *Integrability.* In general, for Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras, the BCH formula is not well-defined. We want to clarify here how to bypass this issue in our context. Let us first recall the formula as presented in [Ser09, Part I, Chapter IV, §7-8]. We focus on the *nilpotent* setting, since it is the only one we will need.

Definition 6.2.7. We say that a Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra $(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$ is *nilpotent* if the underlying Lie algebra $(\mathfrak{a}, [-,-])$ is nilpotent. We also say that a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra is *nilpotent* if the associated Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra is so.

For positive integers d, n, we denote by $\Delta_n(d) \subseteq \mathbb{N}^n \times \mathbb{N}^n$ the subset of elements $(\underline{r}, \underline{s}) \in \mathbb{N}^n \times \mathbb{N}^n$ with $\underline{r} = (r_1, \dots, r_n)$ and $\underline{s} = (s_1, \dots, s_n)$ such that $\sum_{i=1}^n (r_i + s_i) = d$ and $r_i + s_i \neq 0$ for every i. The BCH series can be written as

$$\mathrm{BCH}(X,Y) \coloneqq \sum_{d=1}^{\infty} \sum_{n=1}^{\infty} \sum_{(\underline{r},\underline{s}) \in \Delta_n(d)} (-1)^{n-1} \frac{[X^{r_1}Y^{s_1} \cdots X^{r_n}Y^{s_n}]}{dn \prod_{i=1}^n r_i! s_i!}.$$

If $\mathfrak{a}$ is a nilpotent Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra, then for every $a, b \in \mathfrak{a}$, there is a well-defined element $\mathrm{BCH}(a,b) \in \mathfrak{a}$. Indeed, for d and n big enough the terms of the series vanish. For (Dieudonné–)Lie $\breve{\mathbb{Z}}_p$-algebras, instead, the series might not be defined when the denominators are too divisible by p. It then makes sense to consider the following definition.

Definition 6.2.8. We say that a nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra is *integrable* if for every $a, b \in \mathfrak{a}^+$, each summand

$$(-1)^{n-1}\frac{[a^{r_1}b^{s_1}\cdots a^{r_n}b^{s_n}]}{dn\prod_{i=1}^n r_i!s_i!}$$

of $\mathrm{BCH}(a,b)$ lies in $\mathfrak{a}^+$.

Lemma 6.2.9. *If $\mathfrak{a}^+$ is a nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, then $p^2\mathfrak{a}^+$ is an integrable Dieudonné–Lie $\breve{\mathbb{Z}}_p$-subalgebra.*

Proof. Write $\mathfrak{b}^+$ for the Dieudonné–Lie $\breve{\mathbb{Z}}_p$-subalgebra $p^2\mathfrak{a}^+ \subseteq \mathfrak{a}^+$. We have to prove that for every $v, w \in \mathfrak{b}^+$, each term

$$(-1)^{n-1}\frac{[v^{r_1}w^{s_1}\cdots v^{r_n}w^{s_n}]}{dn\prod_{i=1}^n r_i!s_i!}$$

lies in $\mathfrak{b}^+$. By the bilinearity of the Lie bracket, we have that $[\mathfrak{b}^+, \mathfrak{b}^+] \subseteq p^2\mathfrak{b}^+$. Thus, by induction, we deduce that for every $e \geq 2$ and every set of elements $\{v_1, \ldots, v_e\}$ in $\mathfrak{b}^+$, the nested bracket $[v_1 \cdots v_e]$ lies in $p^{2e-2}\mathfrak{b}^+$. The nested bracket $[v^{r_1}w^{s_1}\cdots v^{r_n}w^{s_n}]$ is then an element of $p^{2d-2}\mathfrak{b}^+$. Since the denominator $dn\prod_{i=1}^n r_i!s_i!$ divides $(d!)^2$, we get the desired result thanks to the fact that $\frac{p^{d-1}}{d!}$ is an element of $\breve{\mathbb{Z}}_p$. $\square$

6.3. **Formal homogeneous spaces.** In this section we focus on the construction of the functor

$$Z\colon \left\{\text{Plain Dieudonné–Lie } \breve{\mathbb{Z}}_p\text{-algebras}\right\} \to \left\{\text{Formal Lie varieties over } \overline{\mathbb{F}}_p\right\}$$

that was defined in [DvH22] to describe the infinitesimal behaviour of central leaves of Shimura varieties. As we will see, $Z(\mathfrak{a}^+)$ admits by construction a transitive action of the formal group $\tilde{\Pi}(\mathfrak{a})$ (Construction 6.3.1). Hence the name *formal homogeneous space attached to* $\mathfrak{a}^+$.

Construction 6.3.1. Let $\mathfrak{a}$ be a nilpotent Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra. The Lie bracket on $\mathfrak{a}$ induces a Lie bracket on $\tilde{\mathbb{X}}(\mathfrak{a})$. The BCH formula defines then a formal group structure[10]

$$m_{\mathrm{Lie}}\colon \tilde{\mathbb{X}}(\mathfrak{a})\times\tilde{\mathbb{X}}(\mathfrak{a}) \to \tilde{\mathbb{X}}(\mathfrak{a})$$

with inverse the multiplication by -1. We define $\tilde{\Pi}(\mathfrak{a})$ to be the formal group $(\tilde{\mathbb{X}}(\mathfrak{a}), m_{\mathrm{Lie}})$. The assignment $\mathfrak{a} \mapsto \tilde{\Pi}(\mathfrak{a})$ defines a functor

$$\tilde{\Pi}\colon \left\{\text{Nilpotent Dieudonné–Lie } \breve{\mathbb{Q}}_p\text{-algebras}\right\} \to \left\{\text{Formal groups over } \overline{\mathbb{F}}_p\right\}.$$

Construction 6.3.2. If $\mathfrak{a}^+$ is an integrable nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, the Lie bracket on $\mathfrak{a}^+$ induces a Lie bracket on $T_p\mathbb{X}(\mathfrak{a}^+)$. The formal group structure m_{Lie} on $\tilde{\mathbb{X}}(\mathfrak{a})$ restricts then

[10]See [*ibid.*,§2.1] for the details on the notion of formal schemes in force.

to a group scheme structure on $T_p\mathbb{X}(\mathfrak{a}^+)$. We denote by $\Pi(\mathfrak{a}^+)$ the group scheme $(T_p\mathbb{X}(\mathfrak{a}^+), m_{\mathrm{Lie}})$. Note that m_{Lie} preserves $p^m T_p\mathbb{X}(\mathfrak{a}^+)$ for every $m \geq 0$, so that

$$\Pi(\mathfrak{a}^+) = \varprojlim_{m\geq 0} \Pi_m(\mathfrak{a}^+),$$

where $\Pi_m(\mathfrak{a}^+)$ is the affine finite scheme $\mathbb{X}(\mathfrak{a}^+)[p^m]$ endowed with the group scheme structure induced by m_{Lie}. This shows that $\Pi(\mathfrak{a}^+)$ is a profinite group scheme.

In this case, the assignment $\mathfrak{a}^+ \mapsto \Pi(\mathfrak{a}^+)$ defines a functor

$$\Pi\colon \left\{\text{Integrable nilpotent Dieudonné–Lie } \breve{\mathbb{Z}}_p\text{-algebras}\right\} \to \left\{\text{Profinite group schemes over } \overline{\mathbb{F}}_p\right\}.$$

Remark 6.3.3. If the slopes of $\mathfrak{a}$ are negative, then $\tilde{\Pi}(\mathfrak{a})$ is a connected affine formal group. Similarly, in the integral situation, if the slopes of $\mathfrak{a}^+$ are negative, then $\Pi(\mathfrak{a}^+)$ is a connected profinite group scheme.

To continue our analysis, it will be convenient to work under additional assumptions on $\mathfrak{a}^+$.

Definition 6.3.4. A *plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra* is a completely slope divisible integrable Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra such that all the slopes are negative.

As we have seen in Construction 6.2.4, for every $\mathfrak{a}^+$ and $n \in \mathbb{Z}$, there is a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra $\Phi^n(\mathfrak{a}^+)$, constructed using the Frobenius, which is isogenous to $\mathfrak{a}^+$. It is easy to check that if $\mathfrak{a}^+$ is plain, even $\Phi^n(\mathfrak{a}^+)$ is plain for every n.

Construction 6.3.5. Let $\mathfrak{a}^+$ be a plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra. For every $n \geq 0$, we write $\Pi^n(\mathfrak{a}^+)$ for $\Pi(\Phi^n(\mathfrak{a}^+))$ and, for $m \geq 0$, we write $\Pi^n_m(\mathfrak{a}^+)$ for $\Pi_m(\Phi^n(\mathfrak{a}^+))$. There are natural maps $\alpha_n\colon \Pi_n(\mathfrak{a}^+) \to \Pi^n_n(\mathfrak{a}^+)$. We define $Z^n(\mathfrak{a}^+)$ to be the fppf-quotient

$$\Pi^n_n(\mathfrak{a}^+)/\alpha_n(\Pi_n(\mathfrak{a}^+))$$

over $\mathrm{Alg}^{\mathrm{op}}_{\overline{\mathbb{F}}_p}$. We also write $Z(\mathfrak{a}^+)$ for the fppf-sheaf obtained as the inductive limit

$$\varinjlim_n Z^n(\mathfrak{a}^+).$$

There is a natural action of $\tilde{\Pi}(\mathfrak{a})$ on the formal scheme $Z(\mathfrak{a}^+)$ and an equivariant map $\tilde{\Pi}(\mathfrak{a}) \to Z(\mathfrak{a}^+)$. We say that $Z(\mathfrak{a}^+)$ is the *formal homogeneous space attached to* $\mathfrak{a}^+$.

Remark 6.3.6. By Lemma 6.2.5, if $\mathfrak{a}^+$ is abelian, then $Z^n(\mathfrak{a}^+) = \mathbb{X}(\mathfrak{a}^+)[F^n]$ and $Z(\mathfrak{a}^+) = \mathbb{X}(\mathfrak{a}^+)$. In general, the formal scheme $Z(\mathfrak{a}^+)$ should be thought as the fpqc-quotient $\tilde{\Pi}(\mathfrak{a})/\Pi(\mathfrak{a}^+)$ (see [DvH22, Lem. 4.3.12]). Note also that in [*ibid.*, Lem. 4.3.11] we prove that $Z(\mathfrak{a}^+)$ satisfies the universal property of a quotient stack over the category of formal schemes.

We want to prove a fundamental representability result for $Z(\mathfrak{a}^+)$. For this we use a construction that allows us to reduce many statements to the case when $\mathfrak{a}^+$ is abelian.

Construction 6.3.7. Let $\mathfrak{a}^+$ be a plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra and let μ_1 be the minimal slope of $\mathfrak{a}^+$. The formal group structure m_{Lie} induces a morphism of affine schemes $\Pi^n_n(\mathfrak{a}^+_{\mu_1}) \times \Pi^n_n(\mathfrak{a}^+) \to \Pi^n_n(\mathfrak{a}^+)$. By Lemma 6.2.6, the group $\tilde{\Pi}(\mathfrak{a}_{\mu_1})$ is in the centre of $\tilde{\Pi}(\mathfrak{a})$, thus we also get a morphism

$$\mathbb{X}(\mathfrak{a}^+_{\mu_1})[F^n] \times Z^n(\mathfrak{a}^+) \to Z^n(\mathfrak{a}^+)$$

which endows $Z^n(\mathfrak{a}^+)$ with a left action of $\mathbb{X}(\mathfrak{a}^+_{\mu_1})[F^n]$. This makes $Z^n(\mathfrak{a}^+)$ an fppf-torsor over $Z^n(\mathfrak{a}^+/\mathfrak{a}^+_{\mu_1})$ under the finite syntomic group scheme $\mathbb{X}(\mathfrak{a}^+_{\mu_1})[F^n]$.

Proposition 6.3.8. *If $\mathfrak{a}^+$ is a plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, then for every $n \geq 1$ the fppf-sheaf $Z^n(\mathfrak{a}^+)$ is represented by* $\mathrm{Spec} R_{n,m}$ *where m is the dimension of $\mathbb{X}(\mathfrak{a}^+)$ and*

$$R_{n,m} \coloneqq \overline{\mathbb{F}}_p[X_1, \cdots, X_m]/(X_1^{p^n}, \cdots, X_m^{p^n}).$$

Moreover, the torsor $Z^n(\mathfrak{a}^+) \to Z^n(\mathfrak{a}^+/\mathfrak{a}^+_{\mu_1})$ of Construction 6.3.7 is trivial.

Proof. We want to prove the result by induction on the number of slopes of $\mathfrak{a}^+$. In the isoclinic case the result follows from Proposition 2.1.2 of [Mes72]. For the inductive step, we first note that by Construction 6.3.7, the fppf-sheaf $Z^n(\mathfrak{a}^+)$ is represented by a connected scheme which is finite and syntomic over $Z^n(\mathfrak{a}^+/\mathfrak{a}^+_{\mu_1})$. The result is then obtained as a consequence of Proposition 3.6.8 of [DG70, Chapter III]. $\square$

Corollary 6.3.9. *If $\mathfrak{a}^+$ is a plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, then $Z(\mathfrak{a}^+)$ is a formal Lie variety of the same dimension of $\mathbb{X}(\mathfrak{a}^+)$.*

By Corollary 6.3.9 we get a functor

$$Z\colon \left\{\text{Plain Dieudonné–Lie } \breve{\mathbb{Z}}_p\text{-algebras}\right\} \to \left\{\text{Formal Lie varieties over } \overline{\mathbb{F}}_p\right\}.$$

Note that the formal schemes $Z(\mathfrak{a}^+)$, besides having a natural action of $\tilde{\Pi}(\mathfrak{a})$, are endowed with a special class of closed formal subschemes given by the following construction.

Construction 6.3.10 (Strongly Tate-linear subspaces)**.** For a Dieudonné–Lie $\breve{\mathbb{Q}}_p$-subalgebra $\mathfrak{b} \subseteq \mathfrak{a}$ we write $\mathfrak{b}^+$ for the intersection $\mathfrak{b} \cap \mathfrak{a}^+$. The inclusion $\mathfrak{b}^+ \subseteq \mathfrak{a}^+$ induces a closed embedding $Z(\mathfrak{b}^+) \subseteq Z(\mathfrak{a}^+)$. Following Chai–Oort, we say that a closed formal subscheme $Z \subseteq Z(\mathfrak{a}^+)$ obtained in this way is a *strongly Tate-linear subspace*, see [Cha20].

Lemma 6.3.11. *The natural map $\tilde{\Pi}(\mathfrak{a}) \to Z(\mathfrak{a}^+)$ is an fpqc-torsor under the affine group scheme $\Pi(\mathfrak{a}^+)$.*

Proof. Thanks to Lemma 6.2.5, for every $n \geq 0$ we have a cartesian diagram

$$\begin{tikzcd} \Pi(\mathfrak{a}^+) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Pi^n(\mathfrak{a}^+) \arrow[d] \\ \Pi_n(\mathfrak{a}^+) \arrow[r] & \Pi^n_n(\mathfrak{a}^+). \end{tikzcd} \tag{6.1}$$

By diagram chasing, we deduce that $\Pi^n(\mathfrak{a}^+) \to Z^n(\mathfrak{a}^+)$ is a torsor for $\Pi(\mathfrak{a}^+)$. This yields the desired result. $\square$

Lemma 6.3.12. *For every plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra $\mathfrak{a}^+$, the natural map $\tilde{\Pi}(\mathfrak{a}) \to Z(\mathfrak{a}^+)$ induces an isomorphism*

$$\tilde{\Pi}(\mathfrak{a}) \simeq Z(\mathfrak{a}^+)^{\mathrm{perf}}.$$

Proof. For every $n \geq 0$, we have the following factorisation of the nth-power of the absolute Frobenius of $Z(\mathfrak{a}^+)$

$$\begin{tikzcd} Z(\mathfrak{a}^+) \arrow[rr, "F^n"] \arrow[dr, "\sim"'] & & Z(\mathfrak{a}^+) \\ & Z(\Phi^{-n}(\mathfrak{a}^+)), \arrow[ur] & \end{tikzcd}$$

where $Z(\Phi^{-n}(\mathfrak{a}^+)) \to Z(\mathfrak{a}^+)$ is induced by the natural inclusion $\Phi^{-n}(\mathfrak{a}^+) \subseteq \mathfrak{a}^+$. This implies that

$$Z(\mathfrak{a}^+)^{\mathrm{perf}} = \varprojlim_n Z(\Phi^{-n}(\mathfrak{a}^+)).$$

Since $\bigcap_n \Phi^{-n}(\mathfrak{a}^+) = 0$, we deduce that $\tilde{\Pi}(\mathfrak{a}) \to \varprojlim_n Z(\Phi^{-n}(\mathfrak{a}^+))$ is a monomorphism, thus by [DvH22, Lem. 2.1.2] it is a closed immersion. On the other hand, $\tilde{\Pi}(\mathfrak{a}) \to \varprojlim_n Z(\Phi^{-n}(\mathfrak{a}^+))$ is flat since for every $n \geq 0$ the map $\tilde{\Pi}(\mathfrak{a}) \to Z(\Phi^{-n}(\mathfrak{a}^+))$ is so. This implies the desired result. □

6.4. **Deformation spaces as formal homogeneous spaces.** For p-divisible groups Y and Z over a perfect field κ, Chai and Oort constructed finite group schemes

$$\mathbf{Hom}^{\mathrm{st}}(Y[p^n], Z[p^n]) \subseteq \mathbf{Hom}(Y[p^n], Z[p^n]),$$

where $\mathbf{Hom}(Y[p^n], Z[p^n])$ is the sheaf of homomorphisms from $Y[p^n]$ to $Z[p^n]$. The natural maps

$$\pi_n : \mathbf{Hom}^{\mathrm{st}}(Y[p^n], Z[p^n]) \to \mathbf{Hom}^{\mathrm{st}}(Y[p^{n+1}], Z[p^{n+1}])$$

make the inductive limit

$$\varinjlim_n \mathbf{Hom}^{\mathrm{st}}(Y[p^n], Z[p^n])$$

a p-divisible group over κ, denoted by $\mathcal{H}_{Y,Z}$. This is called the *internal-Hom p-divisible group* of Y and Z. When $Y = Z$ we denote this p-divisible group by $\mathcal{H}_Y$. If Y is endowed with a G-structure for some reductive group G, one can also define the variant $\mathcal{H}_Y^G \subseteq \mathcal{H}_Y$ looking at endomorphisms which preserve the G-structure (see [DvH22, §4.4]).

Thanks to [CS17, Lem. 4.1.7], the scheme-theoretic p-adic Tate module of $\mathcal{H}_{Y,Z}$ is isomorphic to the group scheme $\mathbf{Hom}(Y, Z)$ of homomorphisms from Y to Z. By [*ibid.*, Lem 4.1.8], there is also an isomorphism

$$\mathbb{D}(\mathcal{H}_{Y,Z})[\tfrac{1}{p}] = \underline{\mathrm{Hom}}(\mathbb{D}(Y)[\tfrac{1}{p}], \mathbb{D}(Z)[\tfrac{1}{p}])_{\leq 0},$$

where $\underline{\mathrm{Hom}}(\mathbb{D}(Y)[\frac{1}{p}], \mathbb{D}(Z)[\frac{1}{p}])$ denotes the internal-Hom in the category of F-isocrystals and $(-)_{\leq 0}$ denotes the operation of taking the subspace of slope ≤ 0. By the proof of Lemma 4.1.8 of [*ibid.*] there is a canonical isomorphism of formal group schemes

$$\widetilde{\mathcal{H}}_{Y,Z} = \mathbf{Hom}(Y, Z)[\tfrac{1}{p}], \tag{6.2}$$

where $\widetilde{\mathcal{H}}_{Y,Z}$ is the universal cover of $\mathcal{H}_{Y,Z}$.

6.4.1. If Y is a completely slope divisible p-divisible group over $\overline{\mathbb{F}}_p$, then the deformation space $\mathbf{Def}_{\mathrm{sus}}(Y)$ of the trivial $\mathbf{Aut}(Y)$-torsor admits a closed immersion into the deformation space $\mathbf{Def}(Y)$ of Y, see [Cha20, Lem. 3.6, Thm. 4.3]. Its image is identified with the subspace of deformations of Y that are fpqc locally isomorphic to the constant deformation of Y. It follows from [Kim19, §5] that there is an action of $\mathbf{Aut}(\tilde{Y})^\circ$ on $\mathbf{Def}_{\mathrm{sus}}(Y)$ and an equivariant map $\mathbf{Aut}(\tilde{Y})^\circ \to \mathbf{Def}_{\mathrm{sus}}(Y)$ which is an $\mathbf{Aut}(Y)^\circ$-torsor in the fpqc topology. Essentially, one can think $\mathbf{Def}_{\mathrm{sus}}(Y)$ as the fpqc quotient

$$\mathbf{Aut}(\tilde{Y})^\circ/\mathbf{Aut}(Y)^\circ.$$

In what follows we want to explain how $\mathbf{Def}_{\mathrm{sus}}(Y)$ is related to the formal homogeneous spaces of §6.3.

Remark 6.4.1. If $Y = Y_1 \oplus Y_2$ has two slopes, then $\mathbf{Aut}(\tilde{Y})^\circ$ is isomorphic to $\widetilde{\mathcal{H}}_{Y_1,Y_2}$ and $\mathbf{Aut}(Y)^\circ \simeq T_p\mathcal{H}_{Y_1,Y_2}$ so that $\mathbf{Def}_{\mathrm{sus}}(Y) \simeq \mathcal{H}_{Y_1,Y_2}$. This gives $\mathbf{Def}_{\mathrm{sus}}(Y)$ the structure of a p-divisible formal group. When Y is ordinary, then $\mathbf{Def}_{\mathrm{sus}}(Y) = \mathbf{Def}(Y)$ and the formal group structure on $\mathbf{Def}(Y)$ is the one coming from the classical Serre–Tate coordinates, see [vH24, §4].

6.4.2. Let $\mathcal{Y}$ be the universal p-divisible group over the sustained deformation space $\mathbf{Def}_{\mathrm{sus}}(Y)$, let $Z \subseteq \mathbf{Def}_{\mathrm{sus}}(Y)$ be a formally smooth closed subscheme, and let $G(\mathcal{M}_Z)$ be the monodromy group[11] of the isocrystal $\mathcal{M} = \mathbb{D}(\mathcal{Y})[\frac{1}{p}]$ restricted to Z. We have a natural inclusion

$$\mathrm{Lie}(G(\mathcal{M}_Z)) \subseteq \mathbb{D}(\mathcal{H}_Y^\circ)[\tfrac{1}{p}] =: \mathfrak{a}_Y$$

of nilpotent Lie algebras. In turn, $\mathfrak{a}_Y$ is naturally a Lie subalgebra of $\mathrm{Lie}(\mathrm{GL}(\mathbb{D}(Y)))[\frac{1}{p}]$. Since $\mathcal{M}$ has the structure of an F-isocrystal we get, by [Cre92a, §2.2], an isomorphism

$$G(\mathcal{M}_Z)^{(p)} \to G(\mathcal{M}_Z),$$

which induces an isomorphism

$$\mathrm{Lie}(G(\mathcal{M}_Z))^{(p)} \to \mathrm{Lie}(G(\mathcal{M}_Z))$$

compatible with the F-structure on $\mathbb{D}(\mathcal{H}_Y^\circ)[\frac{1}{p}]$. In particular $\mathrm{Lie}(G(\mathcal{M}_Z)) \subseteq \mathfrak{a}_Y$ is an F-stable Lie subalgebra.

6.4.3. If we consider $\mathfrak{a}_Y^+ = \mathbb{D}(\mathcal{H}_Y^\circ)$ as a nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, then it is completely slope divisible. Nonetheless, it is generally *not* integrable. For simplicity, in what follows we will assume $\mathfrak{a}_Y^+$ integrable, which happens, for example, when the length of the slope filtration is smaller than the prime p.

The exponential morphism $E\colon \tilde{\mathbb{X}}(\mathfrak{a}_Y) = \tilde{\mathcal{H}}_Y^\circ \to \mathbf{Aut}(\tilde{Y})^\circ$ sending $f \mapsto \sum_{i=0}^\infty f^i/i!$ defines an isomorphism of formal schemes. This isomorphism identifies the formal group law m_{Lie} with the composition group law, so that we get an isomorphism

$$E\colon \tilde{\Pi}(\mathfrak{a}_Y) \xrightarrow{\sim} \mathbf{Aut}(\tilde{Y})^\circ$$

[11] The monodromy group is taken with respect to the closed point of Z. We remove in this section the choice of the point in the notation.

of formal group schemes. Under the assumption that $\mathfrak{a}_Y^+$ is integrable, this isomorphism restricts to an isomorphism of profinite group schemes

$$E\colon \Pi(\mathfrak{a}_Y^+) \xrightarrow{\sim} \mathbf{Aut}(Y)^\circ.$$

Finally, this gives an isomorphism

$$E\colon Z(\mathfrak{a}_Y^+) \xrightarrow{\sim} \mathbf{Def}_{\mathrm{sus}}(Y).$$

What we gained from this isomorphism is that we have now a notion of strongly Tate-linear subspace of $\mathbf{Def}_{\mathrm{sus}}(Y)$ (see §6.3.10). We expect that the monodromy of the restriction of $\mathcal{M}$ to these special subspaces should be determined by the Lie algebra structure of $(\mathfrak{b}, [-,-]) \subseteq (\mathfrak{a}_Y[-,-])$. More precisely, we expect the following to be true.

Conjecture 6.4.2 (D'A–van Hoften)**.** *For every Dieudonné–Lie $\breve{\mathbb{Q}}_p$-subalgebra $\mathfrak{b} \subseteq \mathfrak{a}_Y$, we have*

$$\mathrm{Lie}(G(\mathcal{M}_{Z(\mathfrak{b}^+)})) = \mathfrak{b}.$$

If we denote by $U(\mathfrak{b}) \subseteq \mathrm{GL}(\mathbb{D}(Y))[\frac{1}{p}]$ the unipotent subgroup attached to a nilpotent Lie subalgebra $\mathfrak{b} \subseteq \mathrm{Lie}(\mathrm{GL}(\mathbb{D}(Y)))[\frac{1}{p}]$. Conjecture 6.4.2 is then saying that $G(\mathcal{M}_{Z(\mathfrak{b}^+)}) = U(\mathfrak{b})$.

Example 6.4.3. Let the notation be as in §1.3.1 and consider a central leaf C lying in a Newton stratum associated to a non-central cocharacter ν_b. Take a point $x \in C$ and suppose that the associated p-divisible group Y is completely slope divisible (every Newton stratum admits such a leaf). If $\mathfrak{b}^+ := \mathbb{D}(\mathcal{H}_Y^{G,\circ})$, then $Z(\mathfrak{b}^+) \subseteq \mathbf{Def}_{\mathrm{sus}}(Y)$ can be identified with $C^{/x} \subseteq \mathbf{Def}_{\mathrm{sus}}(Y)$. In this case, by [vH24, Cor. 3.3.5][12], the unipotent radical of the monodromy group $G(\mathcal{M}_C)$ is isomorphic to $U_{\nu_b} = U(\mathfrak{b})$. Thanks to Theorem 5.4.4, we deduce that $U(\mathfrak{b})$ is indeed the monodromy group of $\mathcal{M}$ over $Z(\mathfrak{b}^+)$.

As a special case, if Y is of height h and dimension d, then $\mathbf{Def}_{\mathrm{sus}}(Y)$ can be realised as the formal neighborhood of a central leaf in a PEL type unitary Shimura variety of signature $(h-d, d)$ associated to an imaginary quadratic field E in which p splits. In particular, we know that the monodromy group of $\mathcal{M}$ over $\mathbf{Def}_{\mathrm{sus}}(Y)$ is isomorphic to the unipotent group corresponding to the nilpotent Lie algebra $\mathbb{D}(\mathcal{H}_Y^\circ)[\frac{1}{p}] = \mathfrak{a}_Y$.

6.5. **Boundedness of the monodromy.** In this section we will prove a containment of Conjecture 6.4.2. Let Y be a completely slope divisible p-divisible group over $\overline{\mathbb{F}}_p$ and let $\mathcal{Y}$ be the universal p-divisible group over the sustained deformation space $\mathbf{Def}_{\mathrm{sus}}(Y)$. Let $\mathfrak{a}^+ = \mathbb{D}(\mathcal{H}_Y^\circ)$ be the Dieudonné–Lie algebra associated to the internal-Hom p-divisible group of Y and let $\mathfrak{b} \subseteq \mathfrak{a}$ be a Dieudonné–Lie $\breve{\mathbb{Q}}_p$-subalgebra with associated strongly Tate-linear subspace $Z(\mathfrak{b}^+) \subseteq \mathbf{Def}_{\mathrm{sus}}(Y)$. Write $\mathcal{M} = \mathbb{D}(X)[\frac{1}{p}]$ for the isocrystal over $\mathbf{Def}_{\mathrm{sus}}(Y)$ coming from the Dieudonné module of $\mathcal{Y}$.

Theorem 6.5.1 ([DvH22, Thm. 6.1.1])**.** *There is a natural closed immersion* $G(\mathcal{M}_{Z(\mathfrak{b}^+)}) \hookrightarrow U(\mathfrak{b})$.

In the proof we make use of the Cartier–Witt stacks of Drinfeld [Dri20] and Bhatt–Lurie [BL22] associated to quasi-syntomic schemes of characteristic p. Given such a scheme X, there is a p-adic formal stack $X^{\mathbb{\Delta}}$, the *prismatisation* of X, such that coherent crystals on X are the same as coherent sheaves on $X^{\mathbb{\Delta}}$.

[12]The statement of [vH24, Cor. 3.3.5] contains the assumption that [*ibid.*, Hyp. 2.3.1] holds. This is true for us because K_p is hyperspecial, see [*ibid.* 2.3.2].

Consider the $\mathbf{Aut}(Y)$-torsor

$$\mathbf{Isom}(\mathcal{Y}, Y_{\mathbf{Def}_{\mathrm{sus}}(Y)}) \to \mathbf{Def}_{\mathrm{sus}}(Y). \tag{6.3}$$

The locally free crystal $\mathcal{M}^+ = \mathbb{D}(\mathcal{Y})$ defines a vector bundle $\mathcal{V}^+$ over the prismatisation $\mathbf{Def}_{\mathrm{sus}}(Y)^{\mathbb{\Delta}}$. This vector bundle has an associated frame bundle which we will write suggestively as

$$\mathbf{Isom}(\mathcal{V}^+, \mathbb{D}(Y)_{\mathbf{Def}_{\mathrm{sus}}(Y)^{\mathbb{\Delta}}}). \tag{6.4}$$

If we apply the prismatisation functor to the map (6.3), we get a morphism

$$\mathbf{Isom}(\mathcal{Y}, Y_{\mathbf{Def}_{\mathrm{sus}}(Y)})^{\mathbb{\Delta}} \to \mathbf{Def}_{\mathrm{sus}}(Y)^{\mathbb{\Delta}},$$

which is a torsor under the p-adic formal group[13] $\mathbf{Aut}(Y)^{\mathbb{\Delta}}$. Dieudonné theory gives us a homomorphism of group schemes over $\mathrm{Spf}(\mathbb{Z}_p)$

$$\mathbf{Aut}(Y)^{\mathbb{\Delta}} \to \mathbf{Aut}(\mathbb{D}(Y)) \tag{6.5}$$

and a morphism

$$\mathbf{Isom}(\mathcal{Y}, Y_{\mathbf{Def}_{\mathrm{sus}}(Y)})^{\mathbb{\Delta}} \to \mathbf{Isom}(\mathcal{V}^+, \mathbb{D}(Y)_{\mathbf{Def}_{\mathrm{sus}}(Y)^{\mathbb{\Delta}}}),$$

which is $\mathbf{Aut}(Y)^{\mathbb{\Delta}}$-equivariant via the homomorphism (6.5) (see [DvH22, §6.3.4] for more details). The right hand side roughly speaking parametrises all isomorphisms between $\mathcal{V}^+$ and $\mathbb{D}(Y)_{\mathbf{Def}_{\mathrm{sus}}(Y)^{\mathbb{\Delta}}}$, while the left hand side parametrises those isomorphisms that are compatible with the F-structures.

By construction, after pulling back to $Z(\mathfrak{b}^+)$ the torsor (6.3) has a reduction to a $\Pi(\mathfrak{b}^+)$-torsor. Feeding this fact into the prismatisation machinery gives us a reduction of the torsor (6.4) to a $U(\mathfrak{b}^+)$-torsor. If we apply the Tannakian perspective on torsors and invert p, then this exactly gives us a closed immersion

$$G(\mathcal{M}_{Z(\mathfrak{b}^+)}) \hookrightarrow U(\mathfrak{b}),$$

which is exactly what we wanted to prove.

Remark 6.5.2. Note that the morphism (6.5), constructed in [DvH22, §6.3.3], corresponds in practice to the choice of a $\breve{\mathbb{Z}}_p$-conjugation class of matrices with coefficients in $A_{\mathrm{cris}}(R)$, where $R \coloneqq \Gamma(\mathbf{Aut}(Y), \mathcal{O})$. We wonder whether the entries of these matrices can be related to other constructions of p-adic periods of Y. More in general, under integrability assumptions, we constructed morphisms

$$\Pi(\mathfrak{b}^+)^{\mathbb{\Delta}} \to \mathbf{Aut}(\mathbb{D}(Y)).$$

A better understanding of this kind of representations of $\Pi(\mathfrak{b}^+)^{\mathbb{\Delta}}$ and their relation with the isocrystal $\mathcal{M}_{Z(\mathfrak{b}^+)}$ might lead to the resolution of Conjecture 6.4.2.

[13]Since $\mathbf{Aut}(Y)$ is qrsp this is a formal group and not simply a formal group stack.

7. Monodromy over finite fields and flat Tate conjecture

7.1. **An arithmetic version of the parabolicity conjecture.** Let X be a smooth connected variety over $\mathbb{F}_{p^n}$ with a rational point x. In this case, the fibre functor $\omega_x : \mathbf{Isoc}(X) \to \mathbf{Vec}_{\mathbb{Q}_{p^n}}$ induces a $\mathbb{Q}_{p^n}$-linear fibre functor for the $\mathbb{Q}_{p^n}$-linear Tannakian category $\omega_x : \mathbf{F}^n\text{-}\mathbf{Isoc}(X) \to \mathbf{Vec}_{\mathbb{Q}_{p^n}}$.

Definition 7.1.1. For $(\mathcal{M}, \Phi_{\mathcal{M}}) \in \mathbf{F}^n\text{-}\mathbf{Isoc}(X)$ (resp. $(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}}) \in \mathbf{F}^n\text{-}\mathbf{Isoc}(X)$), we write $G(\mathcal{M}, \Phi_{\mathcal{M}}, x)$ (resp. $G(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}}, x)$) for the algebraic group of automorphisms of the restriction of ω_x to $\langle \mathcal{M}, \Phi_{\mathcal{M}} \rangle$ (resp. $\langle \mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}} \rangle$). The group $G(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}}, x)$ coincides with the arithmetic monodromy group of $(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}})$ defined in [D'Ad20, Def. 3.2.4].

By [AD22, Prop. 2.2.4], for every overconvergent F^n-isocrystal $(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}})$ over X, we have as in Proposition 4.2.9 the following commutative diagram of $\mathbb{Q}_{p^n}$-linear algebraic groups

$$\begin{tikzcd} 1 \arrow[r] & G(\mathcal{M}, x) \arrow[r] \arrow[d, hook] & G(\mathcal{M}, \Phi_{\mathcal{M}}, x) \arrow[r] \arrow[d, hook] & G(\mathcal{M}, \Phi_{\mathcal{M}}, x)^{\mathrm{cst}} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & G(\mathcal{M}^\dagger, x) \arrow[r] & G(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}}, x) \arrow[r] & G(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}}, x)^{\mathrm{cst}} \arrow[r] & 1 \end{tikzcd} \tag{7.1}$$

where the rows are exact. The groups on the right are the Tannaka groups of the category of constant objects in $\langle \mathcal{M}, \Phi_{\mathcal{M}} \rangle$ and $\langle \mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}} \rangle$. We write G for $G(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}}, x)$ and H for $G(\mathcal{M}, \Phi_{\mathcal{M}}, x)$. The slope filtration of $\mathcal{M}_x$ defines a quasi-cocharacter $\lambda : \mathbb{G}_m^{1/\infty} \to G(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}}, x)$. Write $P_G(\lambda) \subseteq G$ for the stabiliser of the slope filtration.

Theorem 7.1.2. *If $(\mathcal{M}, \Phi_{\mathcal{M}})$ has constant slopes, then $H = P_G(\lambda)$. Moreover, if $(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}})$ is semi-simple, H is a parabolic subgroup of G.*

Proof. This follows from Theorem 4.3.2 by arguing as in Proposition 4.4.2. □

Suppose that $(\mathcal{M}, \Phi_{\mathcal{M}})$ has constant slopes. If we write $(\mathcal{N}, \Phi_{\mathcal{N}})$ for $\mathrm{Gr}_{S_\bullet}(\mathcal{M}, \Phi_{\mathcal{M}})$, there is a functor $\langle \mathcal{M}, \Phi_{\mathcal{M}} \rangle \to \langle \mathcal{N}, \Phi_{\mathcal{N}} \rangle$ sending $(\mathcal{M}', \Phi_{\mathcal{M}'})$ to $\mathrm{Gr}_{S_\bullet}(\mathcal{M}', \Phi_{\mathcal{M}'})$. This induces the following commutative diagram with exact rows

$$\begin{tikzcd} 1 \arrow[r] & G(\mathcal{N}, x) \arrow[r] \arrow[d, hook] & G(\mathcal{N}, \Phi_{\mathcal{N}}, x) \arrow[r] \arrow[d, hook] & G(\mathcal{N}, \Phi_{\mathcal{N}}, x)^{\mathrm{cst}} \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & G(\mathcal{M}, x) \arrow[r] & G(\mathcal{M}, \Phi_{\mathcal{M}}, x) \arrow[r] & G(\mathcal{M}, \Phi_{\mathcal{M}}, x)^{\mathrm{cst}} \arrow[r] & 1. \end{tikzcd} \tag{7.2}$$

Even in this case, the natural morphism $G(\mathcal{N}, \Phi_{\mathcal{N}}, x)^{\mathrm{cst}} \to G(\mathcal{M}, \Phi_{\mathcal{M}}, x)^{\mathrm{cst}}$ is an isomorphism since the inclusion $\langle \mathcal{N}, \Phi_{\mathcal{N}} \rangle \hookrightarrow \langle \mathcal{M}, \Phi_{\mathcal{M}} \rangle$ provides an inverse map.

Proposition 7.1.3. *The subgroup $G(\mathcal{N}, \Phi_{\mathcal{N}}, x) \subseteq P_G(\lambda)$ is equal to $Z_G(\lambda)$, the centraliser of the image of λ.*

Proof. By construction, the subgroup $G(\mathcal{N}, \Phi_{\mathcal{N}}, x)$ is in $Z_G(\lambda)$. On the other hand, by Chevalley's theorem, there exists an F^n-isocrystal $(\mathcal{M}', \Phi_{\mathcal{M}'}) \in \langle \mathcal{M}, \Phi_{\mathcal{M}} \rangle$ and a subobject $(\mathcal{L}, \Phi_{\mathcal{L}}) \subseteq$

$\mathrm{Gr}_{S_\bullet}(\mathcal{M}', \Phi_{\mathcal{M}'})$ of rank 1 such that $G(\mathcal{N}, \Phi_{\mathcal{N}}, x)$ is the stabiliser of $L := \omega_x(\mathcal{L}) \subseteq \omega_x(\mathcal{M}') =: V$. Since $(\mathcal{L}, \Phi_{\mathcal{L}})$ has rank 1, it is contained as F^n-isocrystal in the image of the quotient

$$\pi : S_i(\mathcal{M}') \to S_i(\mathcal{M}')/S_{i-1}(\mathcal{M}')$$

for some i. Write $(\widetilde{\mathcal{L}}, \Phi_{\widetilde{\mathcal{L}}}) \subseteq (\mathcal{M}', \Phi_{\mathcal{M}'})$ for $\pi^{-1}(\mathcal{L}, \Phi_{\mathcal{L}})$ and $\widetilde{L} \subseteq V$ for its fibre at x. If s is the slope of $(\mathcal{L}, \Phi_{\mathcal{L}})$ and $V^s \subseteq V$ is the subspace of slope s, then $L = \widetilde{L} \cap V^s$. We deduce that $Z_G(\lambda)$ stabilises L, which gives the desired result. □

Corollary 7.1.4. *The algebraic group $G(\mathcal{N}, \Phi_{\mathcal{N}}, x)^\circ$ contains a Cartan subgroup of $G(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}}, x)^\circ$.*

Proof. If T is a maximal torus of $G(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}}, x)^\circ$ containing the image of λ, the centraliser of T in $G(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}}, x)$ is contained in $Z_G(\lambda) = G(\mathcal{N}, \Phi_{\mathcal{N}}, x)$. This yields the desired result. □

As a result, we deduce the following semi-simplicity result.

Theorem 7.1.5. *Let X be a smooth variety over a finite field $\mathbb{F}_{p^n}$ and $f : A \to X$ an abelian scheme with constant slopes. If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is the F-isocrystal $R^1 f_{\mathrm{cris}*}\mathcal{O}_{A,\mathrm{cris}}$, the induced F-isocrystal $(\mathcal{N}, \Phi_{\mathcal{N}}) := \mathrm{Gr}_{S_\bullet}(\mathcal{M}, \Phi_{\mathcal{M}})$ is semi-simple. In particular, $R^1 f_{\text{ét}*}\underline{\mathbb{Q}_p}$ is a semi-simple lisse $\mathbb{Q}_p$-sheaf over X.*

Proof. By étale descent we may assume that X is connected and admits a rational point x. We may also replace $\Phi_{\mathcal{M}}$ with its n-th power. By [EV24], the F^n-isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ is †-extendable and, by [D'Ad20, Cor. 3.5.2.(ii)], the monodromy group $G(\mathcal{M}^\dagger, x)$ is a reductive group (note that $(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}})$ is pure by the Riemann Hypothesis for abelian varieties). On the other hand, since the action of the p^n-th power Frobenius on the crystalline cohomology groups of A_x is semi-simple (this Frobenius is in the centre of $\mathrm{End}(A_x)$), we get that $G(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}}, x)^{\mathrm{cst}}$ is a reductive group. We deduce by (7.1) that $G(\mathcal{M}^\dagger, \Phi^\dagger_{\mathcal{M}}, x)$ is reductive (see also [Pál22, Thm. 1.2] for a different proof over curves). By Proposition 7.1.3, the group $G(\mathcal{N}, \Phi_{\mathcal{N}}, x)$ is the centraliser of λ in $G(\mathcal{M}, \Phi_{\mathcal{M}}, x)$, thus by [Bor91, Cor. 11.12] it is reductive. This shows that $(\mathcal{N}, \Phi_{\mathcal{N}})$ is semi-simple.

To prove that $R^1 f_{\text{ét}*}\underline{\mathbb{Q}_p}$ is semi-simple it is enough to prove that the associated unit-root F-isocrystal $(\mathcal{V}, \Phi_{\mathcal{V}})$ is semi-simple. This F-isocrystal coincides with the crystalline Dieudonné module of the p-divisible group $A[p^\infty]^{\text{ét}}$. By [BBM82, Thm. 2.5.6.(ii)], the F-isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ is instead the crystalline Dieudonné module of $A[p^\infty]$, thus $(\mathcal{V}, \Phi_{\mathcal{V}})$ is the minimal slope sub-F-isocrystal of $(\mathcal{M}, \Phi_{\mathcal{M}})$. This concludes the proof. □

7.2. **Applications to abelian varieties.** Before the resolution of the parabolicity conjecture, in [AD22], we used the theory of Frobenius tori developed in [D'Ad20] to prove that $G(\mathcal{M}, x) \subseteq G(\mathcal{M}^\dagger, x)$ contained a maximal torus. We used this weaker form of the conjecture to prove a finiteness result for the perfect torsion points of an abelian variety, giving a positive answer to a question by Esnault.

Theorem 7.2.1 ([AD22, Thm. 5.1.1])**.** *Let k be an algebraic closure of a finite field and let E/k be a finitely generated field extension. For every abelian variety A over E such that $\mathrm{Tr}_{E/k}(A) = 0$, the group $A(E^{\mathrm{perf}})_{\mathrm{tors}}$ is finite.*

Thanks to the full parabolicity conjecture, we then proved the following related result as well.

Theorem 7.2.2 ([D'Ad23, Thm. 1.1.3]). *Let $E/\mathbb{F}_p$ be a finitely generated field extension and let A be an abelian variety over E. The group $A(E^{\mathrm{sep}})[p^\infty]$ is finite in the following two cases.*

(i) *If* $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ *is a division algebra.*
(ii) *If* $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ *does not have any factors of Albert's type IV.*

This theorem enriches the list of known results on the finiteness of separable p-torsion points of abelian varieties (see [Vol95] and [Rös20]). It is worth mentioning that abelian varieties with finite separable p-torsion play an important role in the theory of Brauer–Manin obstructions in positive characteristic (see [PV10]).

As a further consequence of the parabolicity conjecture Ambrosi proved the following result.

Theorem 7.2.3 ([Amb23a]). *Let $E/\mathbb{F}_p$ be a finitely generated field extension and let A be a simple abelian variety over E such that $A(E) \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$. The following statements are true.*

(i) $A(E^{\mathrm{perf}})$ is finitely generated if and only if there are no idempotents $0 \neq e \in \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ such that $e(A[p^\infty]^{\text{ét}}) = 0$.
(ii) A is of positive p-rank if and only it the infinitely p-divisible points of $A(E)$ are torsion points.

7.3. **Flat Tate conjecture.** Let X be a smooth projective variety over a finitely generated field k of characteristic p with field of constants κ (defined as the biggest perfect subfield of k). The original version of the Tate conjecture was formulated by Tate using ℓ-adic étale cohomology. For a natural number $r \geq 0$ it has the following form[14].

Conjecture 7.3.1 ($\mathrm{Tate}_\ell(X, r)$). *The cycle class map*

$$\mathrm{CH}^r(X)_{\mathbb{Q}_\ell} \xrightarrow{c_r} H^{2r}_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell(r))^{\Gamma_k}$$

is surjective.

If $r = 1$ this conjecture is independent of ℓ and it is equivalent to the finiteness of the prime-to-p part of the Brauer group of X (see [Tat65] for k finite). When $r = 1$ the conjecture is known, for example, when X is an abelian variety, [Tat66], or a K3 surface.

There is a variant of this conjecture for crystalline cohomology. By [Mor19, Prop. 3.2], for every r the cohomology group $H^{2r}_{\mathrm{cris}}(X)_{\mathbb{Q}_p}$ can be naturally upgraded to an F-isocrystal over k, thus it is endowed with a topologically p-nilpotent connection ∇ and a Frobenius structure. The Tate conjecture in this case has the following form.

Conjecture 7.3.2 ($\mathrm{Tate}_{\mathrm{cris}}(X, r)$). *The cycle class map*

$$\mathrm{CH}^r(X)_{\mathbb{Q}_p} \xrightarrow{c_r} H^{2r}_{\mathrm{cris}}(X)_{\mathbb{Q}_p}^{\nabla=0, \varphi=p^r}$$

is surjective.

[14] Strictly speaking, Tate formulated the conjecture only in codimension 1, thus for $r = 1$.

Pál in [Pál22, Def. 5.3] formulated a variant of the Tate conjecture using rigid cohomology. Because of the knowledge of rigid cohomology at the time, he had to assume the transcendence degree of k to be (at most) one. He first constructed in [*ibid.*, (5.12.2)], using the higher direct image of rigid cohomology of spreading outs, an F-crystal $\mathcal{H}^{2r}_{\mathrm{rig}}(X)$ over κ. Then in [*ibid.*, Prop. 5.19] he constructed cycle class maps

$$\mathrm{CH}^r(X)_{\mathbb{Q}_p} \xrightarrow{c_r} \mathcal{H}^2_{\mathrm{rig}}(X).$$

His constructions can be now generalised to higher transcendence degree thanks to [EV24]. We can then consider the following conjecture

Conjecture 7.3.3 ($\mathrm{Tate}_{\mathrm{rig}}(X, r)$)**.** *The cycle class map*

$$\mathrm{CH}^r(X)_{\mathbb{Q}_p} \xrightarrow{c_r} \mathcal{H}^{2r}_{\mathrm{rig}}(X)^{\varphi = p^r}$$

is surjective.

Thanks to Theorem 3.3.4, we can prove that the crystalline and rigid Tate conjectures are equivalent.

Proposition 7.3.4. *For every smooth projective variety X over k and every $r \geq 0$, the conjectures* $\mathrm{Tate}_{\mathrm{cris}}(X, r)$ *and* $\mathrm{Tate}_{\mathrm{rig}}(X, r)$ *are equivalent.*

Proof. Let $f \colon \mathcal{X} \to S$ a smooth projective spreading out of X, with S a smooth connected scheme over a finite field with generic fibre k. By [Mor19, Prop. 3.2], the higher direct image $R^{2r} f_{\mathrm{cris}*}\mathcal{O}_{\mathcal{X}_{\mathrm{cris}}}$ is an F-isocrystal. By Theorem 3.3.4, we deduce that

$$H^0_{\mathrm{cris}}(S, R^{2r} f_{\mathrm{cris}*}\mathcal{O}_{\mathcal{X}_{\mathrm{cris}}}) = H^{2r}_{\mathrm{cris}}(X)^{\nabla = 0}_{\mathbb{Q}_p}.$$

The result then follows from Kedlaya's full faithfulness, [Ked04b]. □

In codimension 1, Pál proved also that the rigid Tate conjecture and the ℓ-adic étale Tate conjecture are equivalent as well.

Proposition 7.3.5 ([Pál22, Prop. 6.6])**.** *For every smooth projective variety X over k, the conjectures* $\mathrm{Tate}_\ell(X, 1)$ *and* $\mathrm{Tate}_{\mathrm{rig}}(X, 1)$ *are equivalent.*

Both the crystalline and rigid Tate conjectures do not have clear relations with the p-primary torsion of the Brauer group. The Kummer exact sequence gives rather a comparison between the p-torsion of the Brauer with the fppf cohomology of μ_{p^n}. More precisely, we have the following exact sequence

$$0 \to \mathrm{Pic}(X)/p^n \xrightarrow{c_1} H^2_{\mathrm{fppf}}(X, \mu_{p^n}) \to \mathrm{Br}(X)[p^n] \to 0.$$

The first result on the relation between $\mathrm{Tate}_\ell(X, 1)$ and the finiteness of the p-primary torsion of the Brauer was obtained by Milne.

Theorem 7.3.6 ([Mil75])**.** *If X is a smooth projective surface over a finite field such that* $\mathrm{Tate}_\ell(X, 1)$ *is true, then* $\mathrm{Br}(X)[p^\infty]$ *is finite.*

Ulmer in [Ulm14, §7.3.1] considered the following naive p-adic analogue of $\mathrm{Tate}_\ell(X, 1)$ which involves the fppf cohomology of μ_{p^n}.

Conjecture 7.3.7 ($\mathrm{Tate}^{\mathrm{Ulm}}_{\mathrm{fppf}}(X,1)$)**.** *The cycle class map*

$$\mathrm{Pic}(X)_{\mathbb{Q}_p} \xrightarrow{c_1} H^2_{\mathrm{fppf}}(X_{\bar{k}}, \mathbb{Q}_p(1))^{\Gamma_k}$$

is surjective.

Nonetheless, we noticed that Ulmer's conjecture is false in general.

Proposition 7.3.8 ([D'Ad24a, Cor. 6.7])**.** *For every abelian variety B over a finitely generated field k with* $\mathrm{End}(B) = \mathbb{Z}$*, conjecture* $\mathrm{Tate}^{\mathrm{Ulm}}_{\mathrm{fppf}}(B \times B, 1)$ *is false.*

7.3.1. *Relation with the parabolicity conjecture.* Let us try to see more in depth why $\mathrm{Tate}^{\mathrm{Ulm}}_{\mathrm{fppf}}(X,1)$ can not be true in general. Combining the comparison between crystalline and fppf cohomology of $\mathbb{Q}_p(1)$, [[Ill79], Thm. II.5.14], with Theorem 4.1.1 and Theorem 5.2.1, we deduce that the geometric monodromy group of the Galois action on $H^2_{\mathrm{fppf}}(X_{\bar{k}}, \mathbb{Q}_p(1))$ is contained in a Levi subgroup L of the monodromy group[15] $P \coloneqq G(H^2_{\mathrm{cris}}(X)_{\mathbb{Q}_p})$. At the same time, P is the parabolic subgroup of $G \coloneqq G(R^2 f_{\mathrm{rig}*}\mathcal{O}_{\mathcal{X}_{\mathrm{rig}}})$ associated to the slope filtration and by the theory of weights the latter is a reductive group. Thus, there is the extra obstruction given by the (big) unipotent radical of $P = R_u(P) \rtimes L$, which is not captured by $\mathrm{Tate}^{\mathrm{Ulm}}_{\mathrm{fppf}}(X,1)$. Indeed, note that for every G-representation V, we have

$$V^G = V^P = (V^{R_u(P)})^L \subseteq V^L$$

and, in general, the last containment is not an equality.

To overcome this problem we looked instead at the $\mathbb{Q}_p$-vector subspace

$$\left(\varprojlim_n H^2_{\mathrm{fppf}}(X_{\bar{k}}, \mu_{p^n})^k\right)\left[\frac{1}{p}\right] \subseteq H^2_{\mathrm{fppf}}(X_{\bar{k}}, \mathbb{Q}_p(1))^{\Gamma_k}$$

where

$$H^2_{\mathrm{fppf}}(X_{\bar{k}}, \mu_{p^n})^k \coloneqq \mathrm{im}\left(H^2_{\mathrm{fppf}}(X, \mu_{p^n}) \to H^2_{\mathrm{fppf}}(X_{\bar{k}}, \mu_{p^n})\right).$$

The definition of these groups was inspired by the previous definition of the transcendental Brauer group of a variety. We expected that the following property was true.

Conjecture 7.3.9 ($\mathrm{Tate}_{\mathrm{fppf}}(X,1)$)**.** *The cycle class map*

$$\mathrm{Pic}(X)_{\mathbb{Q}_p} \xrightarrow{c_1} \left(\varprojlim_n H^2_{\mathrm{fppf}}(X_{\bar{k}}, \mu_{p^n})^k\right)\left[\frac{1}{p}\right]$$

is surjective.

As an evidence of this claim we proved the conjecture for abelian varieties.

Theorem 7.3.10 ([D'Ad24a])**.** *If X is an abelian variety over k, the conjecture* $\mathrm{Tate}_{\mathrm{fppf}}(X,1)$ *is true.*

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[15] We consider here monodromy groups of isocrystals without Frobenius structure.

We deduced the result by using $\mathrm{Tate}_{\mathrm{cris}}(A\times A,1)$ proved by de Jong, [deJ98, Thm. 2.6]. The main issue that we had to overcome was the lack of a good comparison between crystalline and fppf cohomology of $\mathbb{Z}_p(1)$ over k. In the special case of abelian varieties, we obtain such a comparison in [D'Ad24a, §4] by using the p-divisible group of A. This result was then extended by Lazda and Skorobogatov to K3 surfaces using the Kuga–Satake construction, [LS24].

Note that by Lemma [D'Ad24a, Lem. 3.3], we have a containment

$$H^2_{\mathrm{fppf}}(X_{\bar{k}},\mu_{p^n})^k\subseteq H^0_{\mathrm{fppf}}(k,R^2f_{\mathrm{fppf}*}\mu_{p^n})$$

for every $n\geq 1$. In light of this, we can also consider the following stronger form of $\mathrm{Tate}_{\mathrm{fppf}}(X,1)$.

Conjecture 7.3.11 ($\mathrm{Tate}^{\mathrm{str}}_{\mathrm{fppf}}(X,1)$)**.** *The cycle class map*

$$\mathrm{Pic}(X)_{\mathbb{Q}_p}\xrightarrow{c_1}H^0_{\mathrm{fppf}}(k,R^2f_{\mathrm{fppf}*}\mathbb{Q}_p(1))$$

is surjective.

We expected that both Conjecture 7.3.9 and Conjecture 7.3.11 were equivalent to the ℓ-adic Tate conjecture for divisors. This was proved recently by Li and Qin.

Theorem 7.3.12 ([LQ24])**.** *For every smooth projective variety X over k, both* $\mathrm{Tate}_{\mathrm{fppf}}(X,1)$ *and* $\mathrm{Tate}^{\mathrm{str}}_{\mathrm{fppf}}(X,1)$ *are equivalent to* $\mathrm{Tate}_\ell(X,1)$.

7.4. **The p-primary torsion of the Brauer group.** If k is a finitely generated field extension of $\mathbb{F}_p$ one can not expect $\mathrm{Br}(A)$ to be finite. Indeed, when k is infinite, $\mathrm{Br}(k)$ is an infinite group and it injects into $\mathrm{Br}(A)$. Even $\mathrm{Br}(A)/\mathrm{Br}(k)$ might be infinite since it contains a subgroup isomorphic to $H^1_{\text{ét}}(k,\mathrm{Pic}_{A/k})$. On the other hand, if $\mathrm{Br}(A_{k_s})^k$ is the image of the natural morphism $\mathrm{Br}(A)\to\mathrm{Br}(A_{k_s})$, where k_s is a separable closure of k in an algebraic closure $\bar{k}$, the group $\mathrm{Br}(A_{k_s})^k[\frac{1}{p}]$ is finite by [CS21, Thm. 16.2.3] or [CHT23, Cor. 1.4]. In general, the p-torsion of $\mathrm{Br}(A_{k_s})^k$ is not finite (see [D'Ad24a, Prop. 5.4]). Nonetheless, we proved the following finiteness result.

Theorem 7.4.1 ([D'Ad24a, Thm. 1.1])**.** *Let A be an abelian variety over a finitely generated field k of characteristic $p>0$. The transcendental Brauer group* $\mathrm{Br}(A_{k_s})^k$ *is a direct sum of a finite group and a finite exponent p-group. In addition, if the Witt vector cohomology group $H^2(A_{\bar{k}},W\mathcal{O}_{A_{\bar{k}}})$ is a finite $W(\bar{k})$-module, then* $\mathrm{Br}(A_{k_s})^k$ *is finite.*

The condition on $H^2(A_{\bar{k}},W\mathcal{O}_{A_{\bar{k}}})$ is necessary to remove the "supersingular pathologies" as the one of our counterexample in [D'Ad24a, Prop. 5.4]. It is satisfied, for example, when the p-rank of A is g or $g-1$, where g is the dimension of A (see [Ill83, Cor. 6.3.16]). If the *formal Brauer group* of $A_{\bar{k}}$, denoted by $\hat{\mathrm{Br}}(A_{\bar{k}})$, is a formal Lie group, then by [AM77, Cor. II.4.4] the cohomology group $H^2(A_{\bar{k}},W\mathcal{O}_{A_{\bar{k}}})$ is a finite $W(\bar{k})$-module if and only if $\hat{\mathrm{Br}}(A_{\bar{k}})$ has finite height. Note also that the formal Brauer group of abelian surfaces is always a formal Lie group by [*ibid.*, Cor. II.2.12]. As a consequence of Theorem 4.1.4, we deduced that the subgroup of Galois-fixed points of $\mathrm{Br}(A_{k_s})$, denoted by $\mathrm{Br}(A_{k_s})^{\Gamma_k}$, has finite exponent as well, [D'Ad24a, Cor. 5.3]. This is a variant of [SZ08, Ques. 2] for abelian varieties.

On the other hand, as a consequence of Proposition 7.3.8, we deduced that in general

$$\mathrm{T}_p(\mathrm{Br}(A_{\bar{k}}))^{\Gamma_k}\neq 0.$$

These "exceptional classes" in $\mathrm{T}_p(\mathrm{Br}(A_{\bar{k}}))^{\Gamma_k}$ are naturally related to the specialisation morphism of the Néron–Severi group. We recall the following theorem, which was proved in [And96, Thm. 5.2] in characteristic 0 and in [Amb23b] and [Chr18] in positive characteristic.

Theorem 7.4.2 (André, Ambrosi, Christensen)**.** *Let K be an algebraically closed field which is not an algebraic extension of a finite field, X a finite type K-scheme, and $\mathcal{Y} \to X$ a smooth and proper morphism. For every geometric point $\bar{\eta}$ of X there is an $x \in X(K)$ such that $\mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{Y}_{\bar{\eta}})) = \mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{Y}_x))$.*

As it is well-known, the theorem is false when $K = \bar{\mathbb{F}}_p$ (see [MP12, Rmk. 1.12]). We proved that, in the known counterexamples, the elements in $\mathrm{T}_p(\mathrm{Br}(A_{\bar{k}}))^{\Gamma_k}$ explain the failure of Theorem 7.4.2 when $K = \bar{\mathbb{F}}_p$. More precisely, we proved the following result.

Theorem 7.4.3 ([D'Ad24a, Thm. 1.4])**.** *Let X be an integral normal scheme of finite type over $\mathbb{F}_p$ with generic point $\eta = \mathrm{Spec}(k)$ and let $f : \mathcal{A} \to X$ be an abelian scheme over X with constant slopes. For every closed point $x = \mathrm{Spec}(\kappa)$ of X we have*

$$\mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{A}_{\bar{x}})^{\Gamma_\kappa}) - \mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{A}_{\bar{\eta}})^{\Gamma_k}) \geq \mathrm{rk}_{\mathbb{Z}_p}(\mathrm{T}_p(\mathrm{Br}(\mathcal{A}_{\bar{\eta}}))^{\Gamma_k}).$$

Remark 7.4.4. In the inequality, the left term is "motivic", while the right term comes from some p-adic object which, as far as we know, has no ℓ-adic analogue.

7.4.1. We end this section with an interpretation of Theorem 7.4.3 that uses the point of view of §7.3.1. In this case, we replace the monodromy groups considered there with their arithmetic counterparts, as in §7.1. Thanks to the crystalline Tate conjecture for abelian varieties, we can then read the inequality of Theorem 7.4.3 as the inequality

$$\dim(V^{T_x}) - \dim(V^P) \geq \dim(V^L) - \dim(V^P),$$

where V is $H^2_{\mathrm{cris}}(\mathcal{A}_x)[\frac{1}{p}]$ and $T_x \subseteq L$ is the monodromy group of the Frobenius at x. We wonder whether in Theorem 7.4.3 we can always find a point x such that the inequality becomes an equality. This corresponds to finding a Frobenius torus T_x such that $V^{T_x} = V^L$. This would prove that the group $\mathrm{T}_p(\mathrm{Br}(\mathcal{A}_{\bar{\eta}}))^{\Gamma_k}$ is the "only obstruction" to Theorem 7.4.2 when $K = \overline{\mathbb{F}}_p$.

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