Boundedness of the *p*-primary torsion of the Brauer group of an abelian variety

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Brauer group

Let X be a scheme,

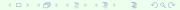
$$\mathrm{Br}(X):=H^2_{\mathrm{\acute{e}t}}(X,\mathbb{G}_{m,X})=H^2_{\mathrm{fppf}}(X,\mathbb{G}_{m,X}).$$

If X is a regular Br(X) is a torsion group.

Let ℓ be a prime number,

$$1 \to \mu_{\ell^n,X} \to \mathbb{G}_{m,X} \xrightarrow{\cdot \ell^n} \mathbb{G}_{m,X} \to 1.$$

$$0 \to \operatorname{Pic}(X)/\ell^n \xrightarrow{c_1} H^2_{\operatorname{fppf}}(X, \mu_{\ell^n, X}) \to \operatorname{Br}(X)[\ell^n] \to 0.$$



Tate conjecture

k finitely generated field $\operatorname{char}(k) = p \geqslant 0$ (e.g. \mathbb{Q} , \mathbb{F}_p , $\mathbb{F}_p(t)$,...).

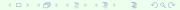
 k_a/k algebraic closure, Γ_k absolute Galois group, X/k smooth proper variety, $\ell \neq p$.

Conjecture (Tate '66)

The cycle class map

$$\operatorname{NS}(X_{k_a})_{\mathbb{Q}_\ell}^{\Gamma_k} \xrightarrow{c_1} H^2_{\operatorname{\acute{e}t}}(X_{k_a}, \mathbb{Q}_\ell(1))^{\Gamma_k}$$

is an isomorphism.



Tate conjecture for abelian varieties

Theorem (Tate, Zarhin, Faltings)

If A is an abelian variety

$$\operatorname{NS}(A_{k_a})_{\mathbb{Z}_\ell}^{\Gamma_k} \xrightarrow{c_1} H^2_{\operatorname{\acute{e}t}}(A_{k_a}, \mathbb{Z}_\ell(1))^{\Gamma_k}$$

is an isomorphism.

Corollary

The group

$$\operatorname{Br}(A_{k_2})[\ell^{\infty}]^{\Gamma_k}$$

is finite for every $\ell \neq p$.



Tate conjecture for abelian varieties

Proof of corollary

$$\operatorname{Br}(A_{k_2})[\ell^{\infty}] \simeq (\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell})^{\oplus (b_2-\rho)}$$

$$\mathrm{T}_{\ell}(\mathrm{Br}(A_{k_a})) := \varprojlim_n \mathrm{Br}(A_{k_a})[\ell^n]$$

$$0 \to \mathrm{NS}(A_{k_a})_{\mathbb{Q}_\ell}^{\Gamma_k} \xrightarrow{c_1} H^2_{\mathrm{\acute{e}t}}(A_{k_a},\mathbb{Q}_\ell(1))^{\Gamma_k} \to \mathrm{T}_\ell(\mathrm{Br}(A_{k_a}))^{\Gamma_k}[\frac{1}{\ell}] \to 0$$

is exact.

 c_1 surjective $\Rightarrow \mathrm{T}_\ell(\mathrm{Br}(A_{k_a}))^{\Gamma_k} = 0 \Rightarrow$ no infinitely ℓ -divisible classes in $\mathrm{Br}(A_{k_a})[\ell^\infty]^{\Gamma_k} \Rightarrow \mathrm{Br}(A_{k_a})[\ell^\infty]^{\Gamma_k}$ is finite.



Transcendental Brauer group

Definition

The transcendental Brauer group of X is

$$\operatorname{Br}(X_{k_a})^k := \operatorname{im}(\operatorname{Br}(X) \to \operatorname{Br}(X_{k_a})) \subseteq \operatorname{Br}(X_{k_a})^{\Gamma_k}.$$

Remark

By the previous corollary, for every $\ell \neq p$, $\operatorname{Br}(A_{k_a})^k[\ell^{\infty}]$ is finite.

What does it happen when $\ell = p > 0$?

Theorem

There exist abelian varieties such that

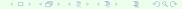
$$\operatorname{NS}(A_{k_a})_{\mathbb{Q}_p}^{\Gamma_k} \xrightarrow{c_1} H^2_{\operatorname{fppf}}(A_{k_a},\mathbb{Q}_p(1))^{\Gamma_k}$$

is not surjective and $T_p(Br(A_{k_n}))^{\Gamma_k} \neq 0$.

We suppose

$$A = B \times B$$

with B an abelian variety.



Leray spectral sequence

We consider
$$E_2^{i,j} := H^i_{\mathrm{fppf}}(B_{k_a}, R^j \pi_{2*} \mu_{p^n}) \Rightarrow H^{i+j}_{\mathrm{fppf}}(A_{k_a}, \mu_{p^n})$$

$$R^1 \pi_{2*} \mu_{p^n} = \mathrm{Pic}_{B_{k_a}/k_a}[p^n] = B^{\vee}_{k_a}[p^n]$$

$$E_2^{1,1} = H^1(B_{k_a}, B^{\vee}_{k}[p^n]) \xrightarrow{h} \mathrm{Hom}(B_{k_a}[p^n], B^{\vee}_{k}[p^n])$$

Construction of h

$$B_{k_a}[p^n] = \operatorname{Hom}(B_{k_a}^{\vee}[p^n], \mathbb{G}_{m,k_a})$$

 $\forall [T] \in H^1(B_{k_a}, B_{k_a}^{\vee}[p^n]) \text{ and } S/k_a \text{ scheme,}$

$$h([T])(S): \operatorname{Hom}(B_{k_a}^{\vee}[p^n], \mathbb{G}_{m,k_a})(S) \to B_{k_a}^{\vee}[p^n](S)$$

$$\tau \mapsto \tau_*(T_S) \in H^1(B_S, \mathbb{G}_{m,B_S})[p^n]$$

Leray spectral sequence

 $R^2\pi_{2*}\mu_{p^n}$ is represented by a linear algebraic group G/k_a .

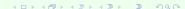
$$\begin{split} E_2^{0,2} &= H^0_{\mathrm{fppf}}(B_{k_a}, R^2 \pi_{2*} \mu_{p^n}) = \mathrm{Mor}(B_{k_a}, G) = \mathrm{Mor}(0_{B_{k_a}}, G) = \\ &= H^2_{\mathrm{fppf}}(B_{k_a}, \mu_{p^n}). \end{split}$$

$$\rightsquigarrow H^2_{\mathrm{fppf}}(A_{k_{\mathsf{a}}},\mu_{p^n}) = H^2_{\mathrm{fppf}}(B_{k_{\mathsf{a}}},\mu_{p^n})^{\oplus 2} \oplus \mathsf{Hom}(B_{k_{\mathsf{a}}}[p^n],B_{k_{\mathsf{a}}}^{\vee}[p^n]).$$

On the other hand,

$$NS(A_{k_a})/p^n = (NS(B_{k_a})/p^n)^{\oplus 2} \oplus Hom(B_{k_a}, B_{k_a}^{\vee})/p^n.$$

 \leadsto Enough to prove that $\operatorname{Hom}(B_{k_a},B_{k_a}^{\vee})_{\mathbb{Q}_p}^{\Gamma_k} \to \operatorname{Hom}(B_{k_a}[p^{\infty}],B_{k_a}^{\vee}[p^{\infty}])[\frac{1}{p}]^{\Gamma_k}$ is not surjective in general.



Counterexample

If B is an elliptic curve with transcendental j-invariant.

$$\mathsf{Hom}(B_{k_a},B_{k_a}^{\vee})_{\mathbb{Q}_p}^{\Gamma_k}=\mathbb{Q}_p$$

$$\operatorname{Hom}(B_{k_a}[p^\infty],B_{k_a}^\vee[p^\infty])[\tfrac{1}{p}]^{\Gamma_k}=\operatorname{Hom}(B_{k_i}[p^\infty],B_{k_i}^\vee[p^\infty])[\tfrac{1}{p}]=\mathbb{Q}_p^{\oplus 2}$$

where $k_i \subseteq k_a$ is the purely inseparable closure.

Conclusion

There is an additional obstruction w.r.t. the purely inseparable extension $k \subset k_i$.

A flat Tate conjecture

$$H^2_{\mathrm{fppf}}(A_{k_{\boldsymbol{a}}},\mu_{\boldsymbol{p}^n})^k := \mathrm{im}(H^2_{\mathrm{fppf}}(A,\mu_{\boldsymbol{p}^n}) \to H^2_{\mathrm{fppf}}(A_{k_{\boldsymbol{a}}},\mu_{\boldsymbol{p}^n})).$$

Theorem

The cycle class map

$$NS(A)_{\mathbb{Z}_p} \xrightarrow{c_1} \varprojlim_n H^2_{\mathrm{fppf}}(A_{k_a}, \mu_{p^n})^k$$

is an isomorphism and $Br(A_k)^k[p^\infty]$ has finite exponent.



Theorem (de Jong '98)

The morphism

$$\operatorname{\mathsf{Hom}}(A,A^{\vee})_{\mathbb{Z}_p} \to \operatorname{\mathsf{Hom}}(A[p^{\infty}],A^{\vee}[p^{\infty}])$$

is an isomorphism.

$$NS(A) \hookrightarrow \mathsf{Hom}^{\mathrm{sym}}(A, A^{\vee})$$

$$[\mathcal{L}] \mapsto \varphi_{\mathcal{L}}$$

 $\varphi_{\mathcal{L}}$ is such that

$$(\mathsf{id} \times \varphi_{\mathcal{L}})^* \mathcal{P}_{A} = \Lambda(\mathcal{L})$$

where \mathcal{P}_A is the Poincaré bundle and $\Lambda(\mathcal{L})$ is the Mumford bundle.



Constructing a morphism

$$\begin{array}{ccc} \operatorname{Pic}(A)^{\bigwedge} & \xrightarrow{c_1} & H^2(A,\mathbb{Z}_p(1)) \\ & & & & & h \end{array}$$

$$\operatorname{Hom}^{\operatorname{sym}}(A,A^{\vee})_{\mathbb{Z}_p} & \xrightarrow{\sim} & \operatorname{Hom}^{\operatorname{sym}}(A[p^{\infty}],A^{\vee}[p^{\infty}]). \end{array}$$

$$h: H^2(A,\mathbb{Z}_p(1)) \xrightarrow{m^* - \pi_1^* - \pi_2^*} F^1H^2(A \times A,\mathbb{Z}_p(1)) \to \operatorname{Hom}(A[p^\infty],A^{\bigvee}[p^\infty])$$

Consequence

Definition

$$H^{2}_{\mathrm{fppf}}(A_{k_{a}}, \mathbb{Z}_{p}(1))^{k} := \mathrm{im}(H^{2}_{\mathrm{fppf}}(A, \mathbb{Z}_{p}(1)) \to H^{2}_{\mathrm{fppf}}(A_{k_{a}}, \mathbb{Z}_{p}(1)))$$
$$\mathrm{T}_{p}(\mathrm{Br}(A_{k_{a}}))^{k} := \mathrm{im}(\mathrm{T}_{p}(\mathrm{Br}(A)) \to \mathrm{T}_{p}(\mathrm{Br}(A_{k_{a}}))).$$

Proposition

The cycle class map

$$\operatorname{NS}(A)_{\mathbb{Z}_p} \xrightarrow{c_1} H^2(A_{k_a},\mathbb{Z}_p(1))^k$$

is surjective and $T_p(Br(A_{k_a}))^k = 0$.

Achtung!

$$T_{p}(Br(A_{k_{2}}))^{k} = T_{p}(Br(A_{k_{2}})^{k}) ?$$

Consequence

Achtung!

$$T_{\rho}(Br(A_{k_a}))^k = T_{\rho}(Br(A_{k_a})^k)$$
 ?

- $\mathrm{Br}(A)[p^n] \twoheadrightarrow \mathrm{Br}(A_{k_a})^k[p^n]$?
- If yes, is the surjectivity preserved by \varprojlim_n ?

$$K_A := \ker(\operatorname{Br}(A) \to \operatorname{Br}(A_{k_a}))$$

$$0 \to \operatorname{Br}(k) \to K_A \to H^1_{\operatorname{finit}}(k, \operatorname{Pic}_{A/k}) \to 0.$$

Solution

Theorem (Gabber, Katz)

There exists a smooth projective connected curve C with $C(k) \neq \emptyset$ endowed with $C \to A$ such that $B := \operatorname{Jac}(C) \twoheadrightarrow A$.

 $B \sim A \times A' \rightsquigarrow$ it is enough to prove the result for B.

F is finite because

$$F = \ker(H^1_{\operatorname{fppf}}(k, \operatorname{Pic}_{B/k}) \to H^1_{\operatorname{fppf}}(k, \operatorname{Pic}_{C/k}))$$

is a subgroup of $H^1_{\text{fronf}}(k, \operatorname{Pic}_{B/k}/\operatorname{Pic}_{B/k}^0)$. Thus $J \subseteq \operatorname{Br}(B)$ provides a splitting of $Br(B) \to Br(B_{k_2})^k$ up to multiplication by m for a fixed $m \geqslant 0$.



Specialisation Néron-Severi group

Theorem (André, Ambrosi, Christensen)

Let K be an algebraically closed field $\neq \overline{\mathbb{F}}_p$, X a finite type K-scheme, and $\mathfrak{Y} \to X$ a smooth and proper morphism. For every geometric point $\overline{\eta}$ of X there is an $x \in X(K)$ such that $\mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathfrak{Y}_{\overline{\eta}})) = \mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathfrak{Y}_{x}))$.

Counterexample

If $A=\mathcal E\times\mathcal E\to X$ where $\mathcal E\to X$ is a non-isotrivial elliptic scheme over $X/\overline{\mathbb F}_p$, then

$$2+2 = \mathsf{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{A}_{\mathsf{x}})) > \mathsf{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{A}_{\overline{\eta}})) = 2+1.$$

Specialisation Néron-Severi group

Theorem

Let X be a connected normal scheme of finite type over \mathbb{F}_p with generic point $\eta = \operatorname{Spec}(k)$ and let $f: \mathcal{A} \to X$ be an abelian scheme over X with constant slopes. For every closed point $x = \operatorname{Spec}(\kappa)$ of X we have

$$\mathsf{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{A}_{\overline{x}})^{\Gamma_{K}}) - \mathsf{rk}_{\mathbb{Z}}(\mathrm{NS}(\mathcal{A}_{\overline{\eta}})^{\Gamma_{k}}) \geqslant \mathsf{rk}_{\mathbb{Z}_{p}}(\mathrm{T}_{p}(\mathrm{Br}(\mathcal{A}_{\overline{\eta}}))^{\Gamma_{k}}).$$

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