Edged crystalline cohomology

Aim: Common framework for crystalline and rigid cohomology, log-decay F-isocrystals, meromorphic flat connections, motives with modulus, ...

R ring

Def An *edged R-algebra* is an algebra obj. (A, F) in exhaustive $\mathbb{Z}$-filtered modules s.t.

$F^0 A \to A$ is
a localisation morphism.

An <u>edged ring</u> is an edged $\mathbb{Z}$-alg.

$(\mathrm{Ring}^{\triangleleft}, \otimes)$: symmetric monoidal category of edged rings

<u>positive</u>:

$$\{\text{Augmented rings}\} \hookrightarrow \mathrm{Ring}^{\triangleleft}$$

$$(A, A/I) \longmapsto \cdots \underset{2}{I^2} \subseteq \underset{1}{I} \subseteq \underset{0}{A} \subseteq \underset{-1}{A} \cdots$$

$$\{\text{Hopf algebras}\} \hookrightarrow \{\text{Hopf edged algebras}\}$$

induced by counit

negative:

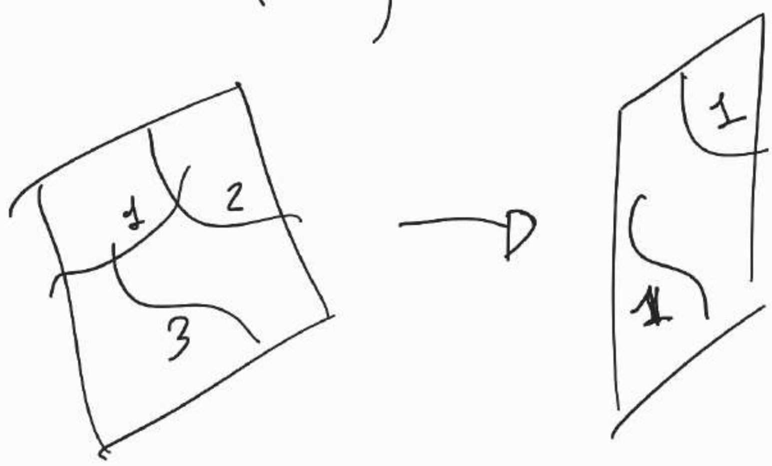
Def A (principal) marked ring is an ordered ring $(A, F^\cdot)$ s.t. $F^\cdot$ is generated by a finite $S \subseteq F^{-1}A \cap A^*$

Ex $f \in B$ nonzero divisor

$$(B, f) \in \mathrm{Ring}^{\triangleleft}$$

$$0 \subseteq \underset{0}{B} \subseteq \underset{-1}{f^{-1}B} \subseteq f^{-2}B \quad \text{—}$$

(modulus pair)



model: $(\mathbb{Z}_p[t]; t)^\wedge \otimes_{\mathbb{Z}_p} (\mathbb{Z}_q, \mathbb{F}_q)$

$$\|$$

$$\ldots \subseteq p\,\mathbb{Z}_p[t, \tfrac{p}{t^n}] \subseteq \mathbb{Z}_p[t, \tfrac{p}{t^m}] \subseteq \tfrac{1}{t^m}\mathbb{Z}_p[t, \tfrac{p}{t^m}] \subseteq \ldots$$

after $(-)^\wedge_p$ in Ring^Δ

$\rightsquigarrow$ Functions converging in

$$p^{-1/m} < |t| \leq 1$$

In general,

$$\tau : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

superadditive, $\neq 0$ (edge-type)

$$(\mathbb{Z}_p[t]; t)^\tau := (\mathbb{Z}_p[t, t^{-1}], F_\tau^\cdot)$$

$$F_\tau^i = \begin{cases} \frac{1}{t^{\tau(i)}} \mathbb{Z}_p[t] & i \geq 0 \\ 0 & i < 0 \end{cases}$$

$$\left(\left(\mathbb{Z}_p[t], t \right)^\tau \hat{\otimes} (\mathbb{Z}_p, \mathbb{F}_p) \right)_p^\wedge$$

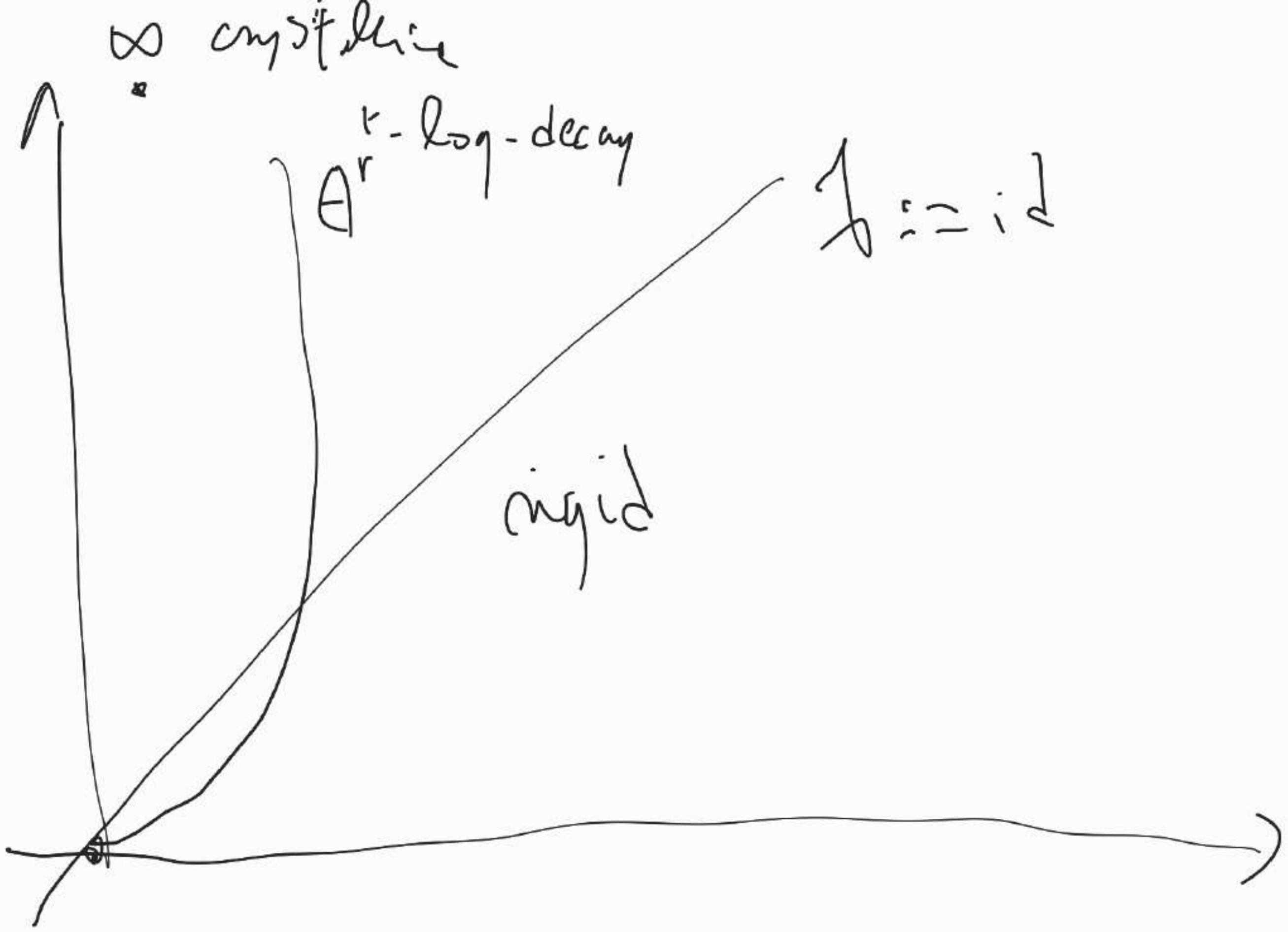
$$\shortparallel$$

"functions with $\hat{\tau}$-decay"

$$\theta_1 : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

$$i \mapsto p^{i-1}$$

corresponds to 1-log-decay functions



Marked crystalline site

Def A *marked* crystalline *extension* is a triple $(A, A/I, \gamma)$ where

- $(A, A/I, \gamma)$ is a crystalline extension

- $\underline{A/I}$ is a marked ring.

$\underline{X}$ marked scheme

$CRIS(\underline{X})$: category of marked crystalline rings

$(A, \underline{A/I}, \gamma)$ over $(\mathbb{Z}_{(p)}, \mathbb{F}_p, \gamma_{can})$

with $\operatorname{Spec}(\underline{A/I}) \to \underline{X}$

$(A, \underline{A/I}, \gamma)$ $\quad$ $\underline{A/I} = (A/I, f)$

$$F^i_{PD} A := \begin{cases} I^{[i]} & i \geq 0 \\ A & i \leq 0 \end{cases}$$

g lift of f, $\pi: A \twoheadrightarrow A/I$

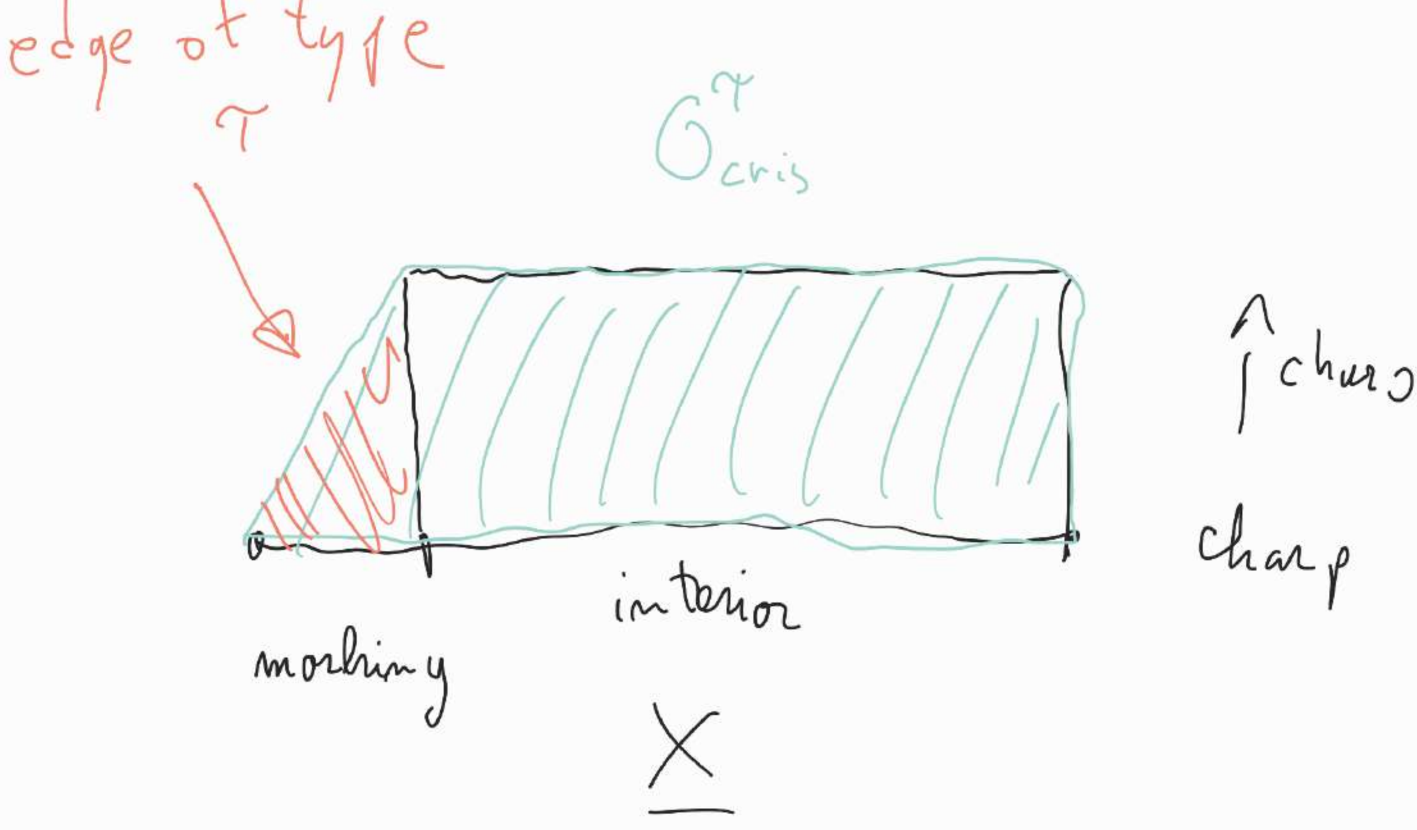
$$\bar{F}^{i}_{\tau} A_g := \begin{cases} \pi^{-1}\left(g^{\tau(i)} A/I\right) & i \geq 0 \\ 0 & i < 0 \end{cases}$$

$$(A, \underline{A/I}, \gamma)^{\tau} := (A, F^{i}_{PD}) \otimes_A (A_g, F^{i}_{\tau})$$

<u>τ-edged localisation</u>

$\rightsquigarrow \mathcal{O}^{\tau}_{cris}$ sheaf of filtered rings

over $CRIS(\underline{X})$



$$\underline{H}^i_{T\text{-cris}}(\underline{X}) := H^i(CRIS(\underline{X}), \mathcal{O}^{\gamma}_{cris})$$

k char p perfect field

X/k separated of finite type (finite expansion)

$$H^i_{T\text{-cris}}(X) := \varinjlim_{X \subseteq \underline{Y}} \underline{H}^i_{T\text{-cris}}(\underline{Y})$$

Y proper (universally closed)

Affine de Rham comparison

$A/_k$ smooth $f \in A$ nonzerodivisor

$\mathcal{A}^\wedge \twoheadrightarrow A$ p-adic lift.

$g \in \mathcal{A}^\wedge$ lift of f.

$$\Omega^{\bullet,+}_{\mathcal{A}^\wedge,f} := (\mathcal{A}^\wedge_g, F^\bullet_{\bar{\imath}}) \otimes (\Omega^\bullet_{\mathcal{A}^\wedge}, F^\bullet_+)$$

with $F^\bullet_+$ generated by $\Omega^1_{\mathcal{A}^\wedge}$ and p at level 1.

Thm If τ is convex, then

$$H^0(\Omega^{\tau,\alpha}_{A,f}) = \underline{H}^0_{\tau\text{-cris}}(\underline{A})$$

Rmk If τ is convex,

$\tau'(e) := \tau(e+1) - \tau(1)$ is convex as well.

Cor If $p \neq 2$ and X admits a smooth compact.

$$H^i_{\tau\text{-cris}}(X)\left[\frac{1}{p}\right] \simeq H^i_{\mathrm{rig}}(X)$$

<u>Rmk</u> For char 0

$(A^\cdot, f) \otimes_{A} (\Omega^\cdot_A, F_{Hdg})$ can be

compared with Deligne's

pole order filtration:

at level $i \leq 0$ we get

$$f^i B \to f^{i-1} \Omega^1_B \to f^{i-2} \Omega^2_B —$$

<u>Conj</u> If $X \subseteq \underline{Y}$ and Y proper, then

$\mathrm{im}\left(\underline{H}^\cdot_{\Delta\text{-cris}}(\underline{Y}) \to H^\cdot_{\Delta\text{-cris}}(X) \right)$

is of finite type.

PD-envelope revisited

$$\mathbb{Z}^{\#} := (\mathbb{Q}, F_{\#}^{\cdot})$$

$$F_{\#}^{i} = \begin{cases} \frac{1}{i!}\mathbb{Z} & i \geq 0 \\ 0 & i < 0 \end{cases}$$

$$\mathbb{Z}^{\#} \underset{\mathbb{Z}}{\otimes} \mathbb{G}_a = \mathbb{G}_a^{\#} \qquad F^0 \mathbb{G}_a^{\#} \simeq \mathrm{Spec}\left(\mathbb{Z}\left[\tfrac{x^n}{n!}\right]\right)$$

$$\mathbb{Z}^{\#} \underset{\mathbb{Z}}{\otimes} \mathbb{G}_m \simeq \mathbb{G}_m^{\#}$$

$$\mathbb{A}^{1,\mathrm{cris}}_{\mathbb{F}_p} = \mathrm{cone}\left(F^0 \mathbb{G}_a^{\#} \to \mathbb{G}_a\right)$$

prismatisation $\mathbb{A}^1_{\mathbb{F}_p}$