

EDGED CRYSTALLINE COHOMOLOGY

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In the memory of Pierre Berthelot

ABSTRACT. We develop a unified foundation for crystalline, convergent, and rigid cohomology through a new family of ringed sites, the τ -edged crystalline sites and τ -edged convergent sites. They are parametrised by certain functions τ called *edge-types*. For linear edge-types, these yield integral versions of rigid cohomology and overconvergent F -isocrystals. For exponential edge-types, the sites realise the conjectured log-decay crystalline cohomology theories, with associated F -isocrystals matching Kramer-Miller's coordinate-defined F -isocrystals. Our approach goes beyond the usual finite-type setting and it is robust enough to support a stacky approach. To analyse rational functions with bounded poles we introduce the notion of *marked schemes*. This formalism offers a convenient framework for studying pole behavior beyond the traditional confines of log geometry.

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Date: August 11, 2025.

2020 Mathematics Subject Classification. 14F30.

Key words and phrases. Crystalline cohomology, convergent cohomology, rigid cohomology, log-decay F -isocrystal, pole order filtration, modulus pair.

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1. INTRODUCTION

In the recent development of p -adic Hodge theory, driven by new techniques such as perfectoid geometry, prismatic cohomology, and Cartier–Witt stacks, convergent and rigid cohomology have remained somehow peripheral to the main advances. This article aims to take a first step towards including Berthelot’s theory within the contemporary landscape of p -adic cohomology theories.

1.1. Crystalline and rigid cohomology. In characteristic p , the two main p -adic cohomology theories are rigid cohomology and crystalline cohomology. Rigid cohomology, constructed by Berthelot, possesses desirable properties such as Poincaré duality, the Lefschetz trace formula, and a theory of weights as for ℓ -adic étale cohomology. However, establishing finiteness results for its cohomology groups remains challenging. Notably, the coherence of the higher direct image of overconvergent F -isocrystals under smooth and proper morphisms, known as *Berthelot’s conjecture*, remains an open problem [Laz16].

On the other hand, crystalline cohomology, pioneered by Grothendieck and chiefly developed by Berthelot and Ogus, is not well behaved for non-proper or singular varieties. Nonetheless, the finiteness statements in the smooth and proper setting are simplified by the comparison with algebraic de Rham cohomology. The crystalline analogue of Berthelot’s conjecture has been solved [Mor19, Appendix], [DTZ23], where the finiteness is established through the coherence of higher direct images of coherent sheaves under proper morphisms. Another significant strength of crystalline cohomology lies in its broader applicability: it is naturally defined for non-Noetherian schemes and it can be used in *perfect geometry*. This often enables reductions to quasiregular semi-perfect schemes, where crystalline cohomology is governed by Fontaine’s ring $\mathbb{A}_{\text{cris}}$. The flexibility of crystalline cohomology allows a stacky approach *à la Simpson*, as developed by Bhatt–Lurie and Drinfeld.

Given the strengths and weaknesses of rigid and crystalline cohomology, the motivation behind edged crystalline cohomology is to amalgamate the desirable aspects of both. Another work along these lines is developed in [Lan23].

1.2. Marked algebraic geometry. To define edged crystalline cohomology, we introduce the notion of *marked schemes*, a generalization of *modulus pairs* defined in [KMSY21]. The theory of marked schemes provides a natural framework for studying regular functions of schemes with poles bounded at the designated marking. With this notion, we are able to construct the *marked infinitesimal site* of a marked scheme $\underline{X}$ over a divided power ring $(A, A/I, \gamma)$. This consists of

infinitesimal extensions $B \rightarrow B/J$ with B/J further endowed with a marking. On this site we construct for every edge-type τ a filtered sheaf of A -algebras $\mathcal{O}_{\text{cris}}^\tau$. This is obtained by taking divided power envelopes, keeping the poles bounded at the marking as dictated by τ .

1.3. Log-decay crystalline cohomology. Wan proved in [Wan96] that the L -function of *log-decay F -isocrystals* with parameter $r \in \mathbb{R}_{>0}$ (i.e., the growth of p -adic valuations of the coefficients is $\frac{1}{r} \log_p$) over $\mathbb{A}_{\mathbb{F}_{p^s}}^n$ is p -adic meromorphic in the open disk of radius $p^{1/r}$. On the other hand, Emerton and Kisin proved in [EK01] that the L -function of unit-root F -isocrystals (corresponding to the limit case $r \rightarrow \infty$) is p -adic meromorphic on the closed disk of radius 1. They employed *Katz's cohomological formula*, formulated with étale cohomology. Wan expected a variant of Katz's cohomological formula, adapted to the log-decay context, that would explain his result. In particular, he expected a *log-decay crystalline cohomology theory* with a suitable Lefschetz trace formula. Our θ^r -edged crystalline cohomology provides a promising candidate for such a theory and could be used to reinterpret and extend his result.

Acknowledgments. It is a pleasure to thank Bruno Chiarellotto, Hélène Esnault, Luc Illusie, Kiran Kedlaya, Shane Kelly, Christopher Lazda, Hiroyasu Miyazaki, Matthew Morrow, Mauro Porta, Kay Rülling, Peter Scholze, and Daqing Wan for all the enlightening discussions on the topic that greatly influenced this article.

The author was funded by the Max-Planck Institute for Mathematics, the Deutsche Forschungsgemeinschaft (project ID: 461915680), and the Marie Skłodowska-Curie Actions (project ID: 101068237). He was also hosted by the Institut de Mathématiques de Jussieu-Paris Rive Gauche and the Institut de Recherche Mathématique Avancée in Strasbourg.

2. MARKED ALGEBRAIC GEOMETRY

The primary goal of this section is to construct the category of *marked schemes*, which generalises the category of *modulus pairs* introduced in [KMSY21].

2.1. Marked rings.

Definition 2.1.1. A *simply marked ring* is a pair (A, I_A^{-1}) with A a ring¹ and $I_A \subseteq A$ an ideal which is invertible² as an A -module. A morphism of simply marked rings $(A, I_A^{-1}) \rightarrow (B, I_B^{-1})$ is a morphism of rings $A \rightarrow B$ such that $I_B \subseteq I_A B$. We denote by Ring^{sim} the (big) category of simply marked rings. If $I_A = (f)$ we also write (A, f^{-1}) for the simply marked ring (A, I_A^{-1}) .

Example 2.1.2. If (A, f^{-1}) and (B, g^{-1}) are simply marked rings with principal ideals, a ring morphism $\varphi: A \rightarrow B$ defines a morphism of simply marked rings if and only if $\varphi(f)|g$. In this case, φ extends naturally to a morphism $A_f \rightarrow B_g$ which sends $\frac{1}{f}A \subseteq A_f$ to $\frac{1}{g}B \subseteq B_g$.

Definition 2.1.3. We denote by Ring the category of commutative rings. There is a natural faithful functor $v: \text{Ring}^{\text{sim}} \rightarrow \text{Ring}$ obtained by forgetting the ideal. This functor admits a fully faithful left adjoint $u: \text{Ring} \rightarrow \text{Ring}^{\text{sim}}$, which sends a ring A to the simply marked ring $(A, 1)$. If not said

¹In this article, by *ring* we always mean a commutative, unital, and associative ring.

²There is a generalisation of the theory where the ideals are not asked to be invertible. The categories we construct in this article should be thought as the “Cartier objects” of larger categories of marked rings and marked schemes.

differently, we use u to compare commutative rings and simply marked rings. We will say that an object in the essential image of u is a *trivially marked ring*.

Lemma 2.1.4. *The category of simply marked rings does not admit coproducts.*

Proof. Suppose that (A, I_A^{-1}) was the coproduct of $(\mathbb{Z}[x], x^{-1})$ and $(\mathbb{Z}[y], y^{-1})$ in $\underline{\text{Ring}}^{\text{sim}}$. If we write $x_A, y_A \in A$ for the images of x and y , we have by definition that the ideal I_A is contained in $(x_A) \cap (y_A)$. By the universal property of (A, I_A^{-1}) , we get the following commutative diagram of simply marked rings

$$\begin{array}{ccc} (A, I_A^{-1}) & \xrightarrow{\varphi_1} & (\mathbb{Z}[x, y], x^{-1}y^{-1}) \\ \downarrow \varphi_2 & & \downarrow \\ (\mathbb{Z}[t], t^{-1}) & \longrightarrow & (\mathbb{Z}[t], t^{-2}), \end{array}$$

where $x \mapsto t$ and $y \mapsto t$. If K is the kernel of $\varphi_1: A \rightarrow \mathbb{Z}[x, y]$, by the previous observation we deduce that

$$I_A + K \subseteq ((x_A) + K) \cap ((y_A) + K) = (x_A y_A) + K.$$

By projecting this containment to $\mathbb{Z}[t]$, this implies that $\varphi_2(I_A) \subseteq (\varphi_2(x_A y_A)) = (t^2)$, contradicting the fact that t lies in $\varphi_2(I_A)$. $\square$

In order to make geometric operations, we want to enlarge the category to allow all finite colimits.

Definition 2.1.5. Let A be a ring and let $\{I_{A,\ell}\}_{\ell \in L}$ be a finite set of invertible ideals of A . The *marking of A associated to $\{I_{A,\ell}\}_{\ell \in L}$* is the functor $h_{\underline{A}}: \underline{\text{Ring}}^{\text{sim}} \rightarrow \text{Set}$ which sends $\underline{R} \in \underline{\text{Ring}}^{\text{sim}}$ to the subset

$$\bigcap_{\ell \in L} \text{Hom}((A, I_{A,\ell}^{-1}), \underline{R}) \subseteq \text{Hom}(A, R).$$

A *marked ring* is the datum of a ring A and a marking $h_{\underline{A}}$ associated to some finite set of invertible ideals of A . A morphism $(A, h_{\underline{A}}) \rightarrow (B, h_{\underline{B}})$ of marked rings is a morphism $A \rightarrow B$ of rings such that the induced morphism³ $h_B \rightarrow h_A$ sends $h_{\underline{B}}(\underline{R})$ to $h_{\underline{A}}(\underline{R})$ for every $\underline{R} \in \underline{\text{Ring}}^{\text{sim}}$. We denote by $\underline{\text{Ring}}$ the category of marked rings. We will see in Lemma 3.2.4 that a marked ring is determined by its *polar filtration*.

Notation 2.1.6. If $(A, h_{\underline{A}})$ is a marked ring associated to the set of invertible ideals $\{I_{A,\ell}\}_{\ell \in L}$, we will also denote it by $(A, I_{A,\ell}^{-1})_{\ell \in L}$. Nonetheless, note that $h_{\underline{A}}$ does not determine uniquely the family $\{I_{A,\ell}\}_{\ell \in L}$, so that $(A, I_{A,\ell}^{-1})_{\ell \in L}$ is only one of many possible presentations of $(A, h_{\underline{A}})$.

Lemma 2.1.7. *If $\underline{A} = (A, I_{A,\ell}^{-1})_{\ell \in L}$ is a marked ring and for some $\ell_1 \neq \ell_2 \in L$ we have $I_{A,\ell_1} \subseteq I_{A,\ell_2}$, then $(A, I_{A,\ell}^{-1})_{\ell \in L} = (A, I_{A,\ell}^{-1})_{\ell \in L \setminus \{\ell_2\}}$. In particular, if A is a valuation ring, then $\underline{A}$ is simply marked.*

Proof. This follows from the fact that for every marked ring $\underline{R} \in \underline{\text{Ring}}^{\text{sim}}$, we have

$$\text{Hom}((A, I_{A,\ell_1}^{-1}), \underline{R}) \subseteq \text{Hom}((A, I_{A,\ell_2}^{-1}), \underline{R}).$$

$\square$

³Note that for rings we use the Yoneda embedding of Ring^{op} rather than Ring .

Definition 2.1.8. We say that a marked ring $\underline{A}$ is *principal* if it can be written in the form $(A, I_{A,\ell}^{-1})_{\ell \in L}$ with each $I_{A,\ell}$ principal. If $\{f_\ell\}_{\ell \in L}$ are generators of $\{I_{A,\ell}\}_{\ell \in L}$ we will also write $(A, f_\ell^{-1})_{\ell \in L}$ rather than $(A, I_{A,\ell}^{-1})_{\ell \in L}$. If $\underline{A} = (A, f_\ell^{-1})_{\ell \in L}$ is a principal marked ring, we say that an element $f \in A$ is a *slicing element* of $\underline{A}$ if

$$\sqrt{(f)} = \sqrt{\prod_{\ell \in L} (f_\ell)}.$$

Construction 2.1.9. Given a ring A and a locally principal ideal I there is a minimal quotient $A \twoheadrightarrow \tilde{A}$ such that $I\tilde{A}$ is invertible in $\tilde{A}$. This corresponds to the ring of global sections of the blowup of $\text{Spec}(A)$ at $V(I)$. If $I = (f)$, then $\tilde{A} = A/K$ where

$$K := \bigcup_{i \geq 1} (0 : (f^i)).$$

Definition 2.1.10. Let $(A, h_{\underline{A}})$ be a marked ring. A *marked envelope* of a ring homomorphism $A \rightarrow B$ is a marked ring $(D, h_{\underline{D}})$ endowed with a morphism $(A, h_{\underline{A}}) \rightarrow (D, h_{\underline{D}})$ such that for every marked ring homomorphism $(A, h_{\underline{A}}) \rightarrow (C, h_{\underline{C}})$ such that $A \rightarrow C$ factors through B , there exists a factorisation

$$(A, h_{\underline{A}}) \rightarrow (D, h_{\underline{D}}) \rightarrow (C, h_{\underline{C}}).$$

Lemma 2.1.11. *For a marked ring $(A, h_{\underline{A}})$ and a ring homomorphism $A \rightarrow B$, there exists a unique marked envelope.*

Proof. If $(A, h_{\underline{A}}) = (A, I_{A,\ell}^{-1})_{\ell \in L}$, then $(D, h_{\underline{D}}) = (\tilde{B}, (I_{A,\ell}\tilde{B})^{-1})_{\ell \in L}$. $\square$

Lemma 2.1.12. *The category $\underline{\text{Ring}}$ has finite colimits and finite products and their formation commutes with the forgetful functor $v: \underline{\text{Ring}} \rightarrow \text{Ring}$. The initial object is $(\mathbb{Z}, 1)$ and the final object is $(0, 0)$.*

Proof. If (A, I_A^{-1}) is a simply marked $(C, 1)$ -algebra and $(B, 1)$ is a trivially marked C -algebra, then the fibre coproduct $(A, I_A^{-1}) \otimes_{(C, 1)} (B, 1)$ is the marked envelope of $A \rightarrow A \otimes_C B$, denoted by (D, I_D^{-1}) . Indeed, for every simply marked ring $\underline{R}$ and every pair

$$(\varphi_1, \varphi_2) \in \text{Hom}(A, R) \times_{\text{Hom}(C, R)} \text{Hom}(B, R)$$

we have

$$(\varphi_1 \otimes \varphi_2)(I_A \otimes B)R = \varphi_1(I_A)R.$$

Therefore, we get

$$\text{Hom}((D, I_D^{-1}), \underline{R}) = \text{Hom}((A, I_A^{-1}), \underline{R}) \times_{\text{Hom}(C, R)} \text{Hom}(B, R).$$

From this we deduce that if $(A, I_{A,\ell}^{-1})_{\ell \in L}$ and $(B, I_{B,m}^{-1})_{m \in M}$ are marked rings over $\text{Spec}(C, I_{A,n}^{-1})_{n \in N}$, the fibre coproduct is the marked ring

$$(D, (I_{A,\ell}D)^{-1}, (I_{B,m}D)^{-1})_{\ell \in L, m \in M}$$

with D the minimal quotient of $A \otimes_C B$ which makes all the ideals invertible. Moreover, if we have a double arrow $f, g: (A, I_{A,\ell}^{-1})_{\ell \in L} \rightarrow (B, I_{B,m}^{-1})_{m \in M}$ and C is the coequaliser at the level of

rings, then the coequaliser as marked rings is the marked envelope of $B \rightarrow C$. The product $(A, I_{A,\ell}^{-1})_{\ell \in L} \times (B, I_{B,m}^{-1})_{m \in M}$ is the marked ring

$$(A \times B, I_{A,\ell}^{-1} \times B, A \times I_{B,m}^{-1}).$$

□

Remark 2.1.13. There is also a nice ∞ -category of *animated marked rings*. One first starts with the ∞ -category $\mathcal{C} = \underline{\text{AnRing}}^{\text{sim}}$ of *animated simply marked rings*. The objects of the homotopy category $\text{h}\mathcal{C}$ are defined to be surjections (at the level of π_0) of animated rings $(A \twoheadrightarrow A_0)$ and the morphisms $(A \twoheadrightarrow A_0) \rightarrow (B \twoheadrightarrow B_0)$ are homotopy classes of morphisms of animated rings $A \rightarrow B$ such that $B \rightarrow B \otimes_A^L A_0$ factors through B_0 (up to homotopy). This category is endowed with a natural functor $v : \text{h}\mathcal{C} \rightarrow \text{hAnRing}$ which sends $(A \twoheadrightarrow A_0)$ to A . Then $\mathcal{C}$ is defined as the fibre product $\text{AnRing} \times_{\text{hAnRing}} \text{h}\mathcal{C}$ as ∞ -categories.

2.2. Marked schemes.

Definition 2.2.1. A *premarked scheme* is a pair $\underline{X} = (X, h_X)$, where X is a scheme and

$$h_X : \underline{\text{Ring}}^{\text{sim}} \rightarrow \text{Set}$$

is a subfunctor of the composition $h_X \circ v$, with $v : \underline{\text{Ring}}^{\text{sim}} \rightarrow \text{Ring}$ the forgetful functor. We say that X is the *underlying scheme* of $\underline{X}$. A morphism of premarked schemes $\underline{X} \rightarrow \underline{Y}$ is a morphism of schemes $X \rightarrow Y$ such that $h_X \rightarrow h_Y$ sends $h_X(\underline{R})$ to $h_Y(\underline{R})$ for every $\underline{R} \in \underline{\text{Ring}}^{\text{sim}}$. We denote by $\underline{\text{Sch}}^\sim$ the (big) category of premarked schemes.

Definition 2.2.2. The category of premarked schemes is naturally fibred over Sch via the faithful functor $v : \underline{\text{Sch}}^\sim \rightarrow \text{Sch}$ which sends $(X, h_X) \mapsto X$. If (X, h_X) is a premarked scheme and $T \rightarrow X$ is a morphism of schemes, we write h_T for $h_X \times_{h_X} h_T$ and we say that a morphism of premarked schemes is *minimal* if it is of the form $(T, h_T) \rightarrow (X, h_X)$.

Definition 2.2.3. There is a canonical fully faithful functor $\underline{\text{Ring}}^{\text{op}} \rightarrow \underline{\text{Sch}}^\sim$ which sends $\underline{A} = (A, h_A)$ to $\text{Spec}(\underline{A}) := (\text{Spec}(A), h_A)$. We say that a premarked scheme isomorphic to $\text{Spec}(\underline{A})$ for some marked ring $\underline{A}$ is an *affine marked scheme*. A *marked scheme* is a premarked scheme (X, h_X) such that for some Zariski covering $\{U_i \rightarrow X\}_{i \in I}$ all the restrictions (U_i, h_{U_i}) are affine marked schemes. We denote by $\underline{\text{Sch}} \subseteq \underline{\text{Sch}}^\sim$ the full (big) subcategory of marked schemes.

A pair $(X, \mathcal{I}_X^{-1})$ where X is a scheme and $\mathcal{I}_X$ is an invertible quasi-coherent sheaf of ideals of X defines a marked scheme. We say that a marked scheme of this form is a *simply marked scheme*. A *principal marked affine scheme* is the spectrum of a principal marked ring. We have a functor $u : \text{Sch} \rightarrow \underline{\text{Sch}}$ sending X to $(X, \mathcal{O}_X)$.

Remark 2.2.4. The notion of modulus pair in [KM21] coincides with the one of a simply marked scheme.

Lemma 2.2.5. *The composition $h_X \circ u$ is represented by an open affine immersion $j : X^\iota \hookrightarrow X$.*

Proof. This can be checked Zariski locally on X . When $\underline{X} = \text{Spec}(A, I_{A,\ell}^{-1})_{\ell \in L}$, we have that

$$X^\iota = \text{Spec}(A) \setminus \bigcup_{\ell \in L} V(I_{A,\ell}).$$

□

Definition 2.2.6. For a marked ring $\underline{A}$ we denote by A^ι the ring $\Gamma((\text{Spec}(\underline{A}))^\iota, \mathcal{O})$. Concretely, when $\underline{A} = (A, f^{-1})$, we have $A^\iota = A_f$.

Lemma 2.2.7. *The category $\underline{\text{Sch}}$ has finite limits and arbitrary coproducts. The final object is $\text{Spec}(\mathbb{Z}, 1)$ and the initial object is $\underline{\emptyset} := \text{Spec}(0, 0)$.*

Proof. This follows from Lemma 2.1.12. □

2.3. Generalities on marked schemes.

Lemma 2.3.1. *Let $(A, h_{\underline{A}})$ be a marked ring and let (B, I_B^{-1}) be a simply marked ring. A morphism $A \rightarrow B$ of rings induces a morphism of marked rings if and only if for every maximal ideal $\mathfrak{m} \subseteq B$, the induced morphisms $(A, h_{\underline{A}}) \rightarrow (B_{\mathfrak{m}}, I_{B_{\mathfrak{m}}}^{-1})$ are morphisms of marked rings. In particular, $h_{\underline{A}}$ is determined by the value on the marked rings $\underline{R} \in \underline{\text{Ring}}^{\text{sim}}$ with R a local ring.*

Proof. It is enough to prove that for every ideal $J \subseteq B$ and $b \in B$ we have that $b \in J$ if and only if $\frac{b}{1} \in JB_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subseteq B$. By replacing B with B/J , we may assume $J = (0)$. Then the result is well-known. □

Construction 2.3.2. When (A, Σ_A^{-1}) be a principal marked ring with $\Sigma_A = \{f_1, \dots, f_n\}$, we write $A[\frac{x}{f_1}, \dots, \frac{x}{f_n}]$ or $A[\frac{x}{\Sigma_A}]$ for the subring of $A^\iota[x]$ generated by $\frac{x}{f_i}$. We have a canonical map

$$(A, \Sigma_A^{-1}) \rightarrow (A[\frac{x}{\Sigma_A}], x^{-1}).$$

Lemma 2.3.3. *Let (A, I_A^{-1}) be a simply marked ring and $(B, g_1^{-1}, \dots, g_n^{-1})$ a principal marked ring. A ring homomorphism $\varphi: A \rightarrow B$ induces a morphism of marked rings if and only if there exist $a_i \in I_A$ and $b_{ij} \in B$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ such that in B^ι we have*

$$\sum_{i,j} \frac{\varphi(a_i)b_{ij}}{g_j} = 1.$$

Proof. Let us first show that the condition is necessary. By definition, the induced morphism

$$(A, I_A^{-1}) \rightarrow (B[\frac{x}{g_1}, \dots, \frac{x}{g_n}], x^{-1})$$

is a morphism of marked rings if and only if there exist $a_1, \dots, a_m \in I_A$ and elements $P_1, \dots, P_m \in B[\frac{x}{g_1}, \dots, \frac{x}{g_n}]$ such that

$$\sum_{i=1}^m \varphi(a_i)P_i = x.$$

Looking at the degree 1 part of this equation, we infer that there exist $b_{ij} \in B$ such that

$$\sum_{i=1}^m \varphi(a_i) \sum_{j=1}^n b_{ij} \frac{x}{g_j} = x.$$

We deduce the desired condition. On the other hand, assuming that the condition is satisfied, we have to show that for every morphism

$$\psi: (B, g_1^{-1}, \dots, g_n^{-1}) \rightarrow (C, h^{-1})$$

of marked rings, the composition $\psi \circ \varphi$ defines a morphism of marked rings. By assumption, for every $1 \leq j \leq n$ there exists $c_j \in C$ such that $\psi(g_j)c_j = h$. This implies that

$$\sum_{i,j} \psi(\varphi(a_i))\psi(b_{ij})c_j = \sum_{i,j} \psi\left(\frac{\varphi(a_i)b_{ij}}{g_j}\right)\psi(g_j)c_j = h,$$

showing that $h \in I_A C$ as we wanted. □

Corollary 2.3.4. *For every principal marked ring (A, Σ_A^{-1}) we have the following canonical isomorphism*

$$(A, \Sigma_A^{-1}) = (A[\frac{x}{\Sigma_A}], x^{-1}) \times_{A^\iota} A.$$

Definition 2.3.5. We denote by $\mathbb{M}$ the marked scheme $\text{Spec}(\mathbb{Z}[t], t^{-1})$. For a marked ring $\underline{A}$, there is a natural inclusion of sets $\mathbb{M}(\underline{A}) \hookrightarrow A$ obtained by taking the image of t in A .

Lemma 2.3.6. *For a marked ring $\underline{A}$ and an element $f \in \mathbb{M}(\underline{A}) \subseteq A$, there exists a unique $g \in A^\iota$ such that $fg = 1$ in A^ι . In other words,*

$$\mathbb{M}(\underline{A}) \subseteq A \cap (A^\iota)^*.$$

Proof. A morphism of marked rings $\varphi : (\mathbb{Z}[t], t^{-1}) \rightarrow \underline{A}$ which sends $t \mapsto f$ induces a morphism $\tilde{\varphi} : \mathbb{Z}[t]_t \rightarrow A^\iota$. We can then take $g := \tilde{\varphi}(\frac{1}{t})$. □

Lemma 2.3.7. *If $\underline{A} = (A, f_1^{-1}, \dots, f_n^{-1})$, then $\mathbb{M}(\underline{A})$ is the set of $g \in A \cap (A^\iota)^*$ such that*

$$\sum_{i=1}^n \frac{a_i}{f_i} = \frac{1}{g}$$

for some $a_i \in A$.

Proof. This follows from Lemma 2.3.3. □

For our purposes, it will be enough to endow the category of marked schemes with a naive Zariski topology. In [D'A25], we also study finer topologies, such as the *v-Zariski topology* and the *v-étale topology*.

Definition 2.3.8. We say that a minimal morphism $\underline{Y} \rightarrow \underline{X}$ is an *open* (resp. *closed*) *immersion* if $Y \rightarrow X$ is an open (resp. closed) immersion.

Definition 2.3.9. A *Zariski covering* of $\underline{X}$ is a family $\{\underline{U}_i \rightarrow \underline{X}\}_{i \in I}$ of open immersions such that $\{U_i \rightarrow X\}_{i \in I}$ is a Zariski covering. We endow $\underline{\text{Sch}}$ with the topology induced by the Zariski coverings. We call it the *Zariski topology*.

Construction 2.3.10. The structural sheaf $\mathcal{O}$ is clearly a Zariski sheaf. More in general, every representable presheaf of $\underline{\text{Sch}}$ is a Zariski sheaf by Lemma 2.3.1. Another example is the sheaf $\mathcal{O}^\iota(\underline{X}) := \Gamma(X^\iota, \mathcal{O})$.

3. FILTRATIONS

This section serves as an interlude on filtrations. We reinterpret the notion of marked rings and marked schemes as filtered objects.

3.1. Filtered objects.

Definition 3.1.1 (Filtered object). Let $\mathcal{A}$ be an abelian category satisfying condition (AB5). A *(descending exhaustive) filtration* Fil of an object $A \in \mathcal{A}$ is the datum of a set $\{\text{Fil}^i(A)\}_{i \in \mathbb{Z}}$ of subobjects of A such that $\text{Fil}^i(A) \subseteq \text{Fil}^j(A)$ for $i \geq j$ and

$$\varinjlim_i \text{Fil}^i(A) = A.$$

We use the convention that $\text{Fil}^{-\infty}(A) = A$. A filtration is *positive* if $\text{Fil}^i(A) = A$ for $i < 0$ and *negative* if $\text{Fil}^i(A) = 0$ for $i > 0$. Given a filtration Fil of an object A and $j \in \mathbb{Z}$, we write $\text{Fil}\langle j \rangle$ for the filtration defined by

$$\text{Fil}\langle j \rangle^i(A) := \text{Fil}^{i+j}(A).$$

A *filtered object* is an object of $\mathcal{A}$ endowed with a filtration. We denote by $\text{Fil}(\mathcal{A})$ the additive category of filtered objects of $\mathcal{A}$. There is a natural embedding $\mathcal{A} \hookrightarrow \text{Fil}(\mathcal{A})$ obtained by endowing $A \in \mathcal{A}$ with the *trivial filtration* Fil_{triv} defined by

$$\text{Fil}_{\text{triv}}^i(A) := \begin{cases} 0 & \text{if } i < 0 \\ A & \text{if } i \geq 0. \end{cases}$$

For $j \in \mathbb{Z}$ we denote by $A\langle j \rangle$ the filtered object $(A, \text{Fil}_{\text{triv}}\langle j \rangle)$.

Definition 3.1.2. Recall that an additive category $\mathcal{C}$ is *quasi-abelian* if the following conditions are satisfied.

- (1) $\mathcal{C}$ admits kernels and cokernels.
- (2) The pushout of a strict epimorphism along arbitrary morphisms is strict.
- (3) The pullback of a strict monomorphism along arbitrary morphisms is strict.

Theorem 3.1.3 ([SS13, Thm 3.9]). *The category $\text{Fil}(\mathcal{A})$ is quasi-abelian.*

Write $\mathcal{A}^{\mathbb{Z}}$ for the category $\text{Fun}((\mathbb{Z}, \geq)^{\text{op}}, \mathcal{A})$ where $(\mathbb{Z}, \geq)$ is $\mathbb{Z}$ endowed with the standard poset structure. Schapira and Schneiders prove that the natural functor

$$\iota: \text{Fil}(\mathcal{A}) \rightarrow \mathcal{A}^{\mathbb{Z}}$$

admits a left adjoint κ which sends $\{A_i\}_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ to $A := \varinjlim_i A_i$ endowed with the filtration

$$\text{Fil}^i(A) := \text{im}(A_i \rightarrow A).$$

In addition, they prove that ι is fully faithful and strictly exact and $\text{Fil}(\mathcal{A})$, via this embedding, is closed under the operation of taking subobjects. In particular, one can check whether a sequence $\text{Fil}(\mathcal{A})$ is strict by passing to $\mathcal{A}^{\mathbb{Z}}$.

Definition 3.1.4. If $\mathcal{A}$ is further endowed with an additive monoidal structure $\otimes$ which commutes with filtered colimits, then $\text{Fil}(\mathcal{A})$ is endowed with a natural additive monoidal structure

$$(A, \text{Fil}) \otimes (B, \text{Fil}) := (A \otimes B, \text{Fil}),$$

with $\text{Fil}^n(A \otimes B)$ generated by the images of $\text{Fil}^i(A) \otimes \text{Fil}^j(B)$ for $i + j = n$.

Definition 3.1.5 (Filtered algebra). If R is a ring and $\text{Mod}(R)$ is the abelian symmetric monoidal category of R -modules, we say that a commutative monoid object of $\text{Fil}(\text{Mod}(R))$ is a *filtered R -algebra*. We denote by $\text{FilAlg}(R)$ the category they form. A *filtered ring* is a commutative filtered $\mathbb{Z}$ -algebra and we write FilRing for $\text{FilAlg}(\mathbb{Z})$. Concretely, a filtered ring is a ring A endowed with a descending multiplicative filtration Fil such that $1 \in \text{Fil}^0(A)$. A *module over a filtered ring* (A, Fil) is a module object over (A, Fil) in $\text{Fil}(\text{Ab})$.

Lemma 3.1.6. *The forgetful functor $\text{FilAlg}(R) \rightarrow \text{Fil}(\text{Mod}(R))$ admits a left adjoint*

$$(M, \text{Fil}) \mapsto \text{Sym}_R(M, \text{Fil}).$$

Proof. The underlying algebra of $\text{Sym}_R(M, \text{Fil})$ is the symmetric tensor algebra

$$\text{Sym}_R(M) := \bigoplus_{i=0}^{\infty} \text{Sym}_R^i(M).$$

The filtration is the one induced by Fil on $M^{\otimes i}$. □

Definition 3.1.7. We say that $\text{Sym}_R(M, \text{Fil})$ is the *filtered symmetric tensor algebra* associated to (M, Fil) .

Lemma 3.1.8. *If (A, Fil) is a filtered R -module and A has in addition an R -algebra structure, then there exists a minimal filtration Fil_{mult} on A coarser than Fil such that $(A, \text{Fil}_{\text{mult}})$ is a filtered R -algebra.*

Proof. There is a natural quotient map $\text{Sym}_R(A) \twoheadrightarrow A$ of R -algebras. The filtration of $\text{Sym}_R(A, \text{Fil})$ induces then a filtration Fil_{mult} on A that satisfies the desired property. □

Definition 3.1.9. For a filtered sheaf of abelian groups $(\mathcal{F}, \text{Fil})$ over a site $\mathcal{C}$, we define $R\Gamma(\mathcal{C}, (\mathcal{F}, \text{Fil}))$ as the object

$$i \mapsto R\Gamma(\mathcal{C}, \text{Fil}^i(\mathcal{F}))$$

in $D(\text{Ab}^{\mathbb{Z}})$.

Lemma 3.1.10. *For a filtered ring (A, Fil) and a filtered (A, Fil) -module (M, Fil) , there exists a strict exact sequence*

$$\bigoplus_{i \in I} (A, \text{Fil}) \rightarrow \bigoplus_{j \in J} (A, \text{Fil}) \rightarrow (M, \text{Fil}) \rightarrow 0$$

for some sets I, J .

3.2. Polar filtration of marked schemes. We want to construct a canonical negative filtration on marked rings.

Definition 3.2.1. For a principal marked ring $\underline{A}$ the associated *polar filtration* Fil_{pol} is the smallest negative filtration on A^{ι} such that $\text{Fil}_{\text{pol}}^0(A^{\iota}) = A$ and for every $f \in \mathbb{M}(\underline{A})$, we have $f^{-1} \in \text{Fil}_{\text{pol}}^{-1}(A^{\iota})$.

Lemma 3.2.2. *For a principal marked ring $(A, f_1^{-1}, \dots, f_n^{-1})$ and a multiplicative subset $S \subseteq A$, we have*

$$S^{-1}(\text{Fil}_{\text{pol}}^i(A^{\iota})) = \text{Fil}_{\text{pol}}^i((S^{-1}A)^{\iota})$$

for every i , where $S^{-1}A$ is endowed with the marking induced by $f_1^{-1}, \dots, f_n^{-1}$.

Proof. This follows from Lemma 2.3.7. $\square$

Definition 3.2.3. By gluing the filtration Fil_{pol} on principal marked rings using Lemma 3.2.2, we get a negative filtration Fil_{pol} for every marked ring called the *polar filtration*.

Lemma 3.2.4. *The functor*

$$\underline{\text{Ring}} \hookrightarrow \text{FilRing}$$

which sends $\underline{A}$ to $(A^\iota, \text{Fil}_{\text{pol}})$ is fully faithful.

Proof. By going Zariski locally it is enough to prove that for every principal simply marked ring (A, f^{-1}) and every principal marked ring $(B, g_1^{-1}, \dots, g_m^{-1})$, a morphism $\varphi: A \rightarrow B$ upgrades to a morphism of marked rings if and only if it induces a morphism $\text{Fil}_{\text{pol}}^{-1}(A^\iota) \rightarrow \text{Fil}_{\text{pol}}^{-1}(B^\iota)$. By Lemma 2.3.3, we already know that φ defines a morphism of marked rings if and only if there exist $b_i \in B$ such that

$$\sum_{i=1}^m \frac{\varphi(f)b_i}{g_i} = 1.$$

Dividing by $\varphi(f)$ this is precisely the condition

$$\varphi\left(\frac{1}{f}\right) \in \text{Fil}_{\text{pol}}^{-1}(B^\iota),$$

as we wanted. $\square$

Remark 3.2.5. Lemma 3.2.4 can be used to redefine marked rings as the essential image of the functor.

Definition 3.2.6. A *filtered scheme* is a scheme X endowed with a sheaf of commutative filtered $\mathcal{O}_X$ -algebras $(\mathcal{O}_X^\iota, \text{Fil})$, such that the unit $\mathcal{O}_X \rightarrow \mathcal{O}_X^\iota$ induces an isomorphism $\mathcal{O}_X \xrightarrow{\sim} \text{Fil}^0(\mathcal{O}_X^\iota)$. We write $X^\iota \rightarrow X$ for the affine morphism induced by $\mathcal{O}_X \rightarrow \mathcal{O}_X^\iota$. We write FSch for the category they form.

Lemma 3.2.7. *There exists a natural fully faithful functor*

$$\underline{\text{Sch}} \hookrightarrow \text{FilSch}.$$

3.3. Filtered differential graded algebras.

Definition 3.3.1 (Filtered differential graded algebra). For a ring R , we write DGM_R for the abelian category of *differential graded R -modules*, namely the category of $\mathbb{Z}_{\geq 0}$ -graded modules with an R -linear endomorphism of degree 1. It is endowed with the usual symmetric monoidal structure such that the braiding $M \otimes_R N \xrightarrow{\sim} N \otimes_R M$ for homogeneous modules M, N is given by

$$m \otimes n \mapsto (-1)^{\deg(m)\deg(n)} n \otimes m.$$

A commutative monoid object in DGM_R is a *differential graded algebra over R* . A *filtered differential graded R -algebra* is instead a commutative monoid object in $\text{Fil}(\text{DGM}_R)$. We write FDGA_R for the category they form.

Definition 3.3.2 (Hodge differential graded algebras). A *Hodge differential graded algebra over R* is a filtered differential graded algebra such that

$$(3.1) \quad d(\text{Fil}^i(A^\iota)) \subseteq \text{Fil}^{i+1}(A^{i+1}).$$

for $i \geq 0$. We call (3.1) the *Hodge condition*. We write HDGA_R for the category of Hodge differential graded algebras over R .

Remark 3.3.3. If A is an R -algebra, then the de Rham complex $\Omega_{A/R}^\bullet$ endowed with the Hodge filtration is the most basic example of a Hodge differential graded algebra over R .

Lemma 3.3.4. *The natural inclusion*

$$\mathrm{HDGA}_R \hookrightarrow \mathrm{FDGA}_R$$

admits a left adjoint

$$\mathrm{Hdg}: \mathrm{FDGA}_R \rightarrow \mathrm{HDGA}_R.$$

Construction 3.3.5 (Hodge filtration). Let (A, Fil) be a filtered ring over a base ring R . The *naive filtration* $\mathrm{Fil}_{\mathrm{naive}}$ on $\Omega_{A/R}^1$ is the filtration induced by the one of A by considering the tensor product $(A, \mathrm{Fil}) \otimes_{A(0)} \Omega_{A/R}^1(0)$. This filtration extends to a filtration $\mathrm{Fil}_{\mathrm{naive}}$ on the differential graded algebra $\Omega_{A/R}^\bullet$. Nonetheless, $(\Omega_{A/R}^\bullet, \mathrm{Fil}_{\mathrm{naive}})$ in general fails to satisfy the Hodge condition. The filtration that we consider is instead

$$(\Omega_{A/R}^\bullet, \mathrm{Fil}_{\mathrm{Hdg}}) := \mathrm{Hdg}(\Omega_{A/R}^\bullet, \mathrm{Fil}_{\mathrm{naive}}).$$

We say that $\mathrm{Fil}_{\mathrm{Hdg}}$ is the *Hodge filtration* of the de Rham complex of a filtered ring. We will also denote $(\Omega_{A/R}^\bullet, \mathrm{Fil}_{\mathrm{Hdg}})$ by $\Omega_{(A, \mathrm{Fil})/R}^\bullet$.

4. THE EDGED CRYSTALLINE SITE

4.1. Marked infinitesimal extensions.

Definition 4.1.1. A *marked infinitesimal extension* $(A, A/I)$ is the datum of a ring quotient $A \twoheadrightarrow A/I$ with nilpotent kernel endowed with a marking on A/I .

Definition 4.1.2. Let $(A, A/I)$ be a ring quotient and $\underline{X}$ a marked scheme over A/I . We write $\mathrm{INF}(\underline{X}/A)$ for the opposite of the category of marked infinitesimal extensions $(B, B/J)$ over $(A, A/I)$ endowed with a morphism $\mathrm{Spec}(B/J) \rightarrow \underline{X}$ over A/I . We denote by $(\underline{X}/A)_{\mathrm{inf}} \subseteq \mathrm{INF}(\underline{X}/A)$ the full subcategory of those $(B, B/J)$ such that $\mathrm{Spec}(B/J) \rightarrow \underline{X}$ is an open immersion.

Definition 4.1.3. A family of morphisms $\{(B_k, B_k/J_k) \rightarrow (B, B/J)\}_{k \in K}$ in $\mathrm{INF}(\underline{X}/A)$ is a *Zariski covering* if the following conditions are satisfied.

- (1) $J_k = JB_k$ for every k .
- (2) $\{\mathrm{Spec}(B_k) \rightarrow \mathrm{Spec}(B)\}_{k \in K}$ is a Zariski covering.
- (3) for every $k \in K$ the morphism $\mathrm{Spec}(B_k/J_k) \rightarrow \mathrm{Spec}(B/J)$ is minimal.

The category $\mathrm{INF}(\underline{X}/A)$ endowed with the Zariski topology is the *(big) marked infinitesimal site*.

Definition 4.1.4. We write $\mathcal{O}_{\mathrm{inf}}$ for the sheaf of A -algebras over $\mathrm{INF}(\underline{X}/A)$ given by $(B, B/J) \mapsto B$ and $\mathcal{I}_{\mathrm{inf}} \subseteq \mathcal{O}_{\mathrm{inf}}$ for the subsheaf $(B, B/J) \mapsto J$. We also write $\mathbb{M}_{\mathrm{inf}}$ for $(\mathbb{Z}[t], (\mathbb{Z}[t], t^{-1})) \in \mathrm{INF}(\underline{X}/A)$.

Remark 4.1.5. The functoriality of the marked infinitesimal topos can be obtained as in [BS22, Rmk. 4.3]

4.2. Edged localisation.

Construction 4.2.1. The sheaf $\mathcal{O}_{\text{inf}}$ is endowed with the *infinitesimal filtration* given by

$$\text{Fil}_{\text{inf}}^i(\mathcal{O}_{\text{inf}}) := \mathcal{I}_{\text{inf}}^i.$$

For every infinitesimal extension $(B, \underline{B/J})$, the topological spaces of $\text{Spec}(B)$ and $\text{Spec}(B/J)$ coincide. Therefore, there exists an open subscheme $U \subseteq \text{Spec}(B)$ corresponding to the open $\text{Spec}(B/J)^\iota \subseteq \text{Spec}(B/J)$. Locally on $\text{Spec}(B)$, the open U is obtained by inverting finitely many functions. We deduce that U is affine and we write B^ι for $\Gamma(U, \mathcal{O}_U)$. We denote by $\mathcal{O}_{\text{inf}}^\iota$ the sheaf over $\text{INF}(\underline{X}/A)$ given by $(B, \underline{B/J}) \mapsto B^\iota$. We denote by $\tilde{\mathcal{O}}_{\text{inf}} \subseteq \mathcal{O}_{\text{inf}}^\iota$ the image of $\mathcal{O}_{\text{inf}} \rightarrow \mathcal{O}_{\text{inf}}^\iota$. The sheaf $\mathcal{O}_{\text{inf}}^\iota$ is endowed with a natural negative filtration with $\text{Fil}_{\text{pol}}^0(\mathcal{O}_{\text{inf}}^\iota) := \tilde{\mathcal{O}}_{\text{inf}}$ and $\text{Fil}_{\text{pol}}^{-1}(\mathcal{O}_{\text{inf}}^\iota)$ locally generated by the inverses of the sections of $\mathbb{M}_{\text{inf}}$. The filtration is called the *polar filtration*. The sheaf $\mathcal{O}_{\text{inf}}^\iota$ is also endowed with an infinitesimal positive filtration such that $\text{Fil}_{\text{inf}}^e(\mathcal{O}_{\text{inf}}^\iota) := \mathcal{I}_{\text{inf}}^e \mathcal{O}_{\text{inf}}^\iota$.

Definition 4.2.2. An *edge-type* is a map of sets $\tau : \mathbb{Z}_{>0} \rightarrow \mathbb{N}_\infty := \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that $\tau \not\equiv 0$ and

$$\tau(e_1 + e_2) \geq \tau(e_1) + \tau(e_2)$$

for every $e_1, e_2 \in \mathbb{Z}_{>0}$. For an edge-type τ and a positive integer n , we denote by $\tau_n : \mathbb{Z}_{>0} \rightarrow \mathbb{N}_\infty$ the *n-th scaling* of τ , defined by $\tau_n(e) = n\tau(e)$, and by $\tau[n]$ the *n-th shift* of τ , defined by $\tau[n](e) = \tau(e + n) - \tau(n)$.

Definition 4.2.3. For an edge-type τ and an object A of an abelian category endowed with a negative filtration Fil , the τ -*dilation* of Fil , denoted by $\tau \star \text{Fil}$, is the negative filtration of A defined by

$$\tau \star \text{Fil}^{-i} := \text{Fil}^{-\tau(i)}$$

for $i \geq 0$.

Definition 4.2.4. We denote by $\mathcal{O}_{\text{inf}}^\tau = (\mathcal{O}_{\text{inf}}^\iota, \text{Fil}_\tau)$ the sheaf of filtered rings

$$\mathcal{O}_{\text{inf}}^\tau := (\mathcal{O}_{\text{inf}}^\iota, \tau \star F_{\text{pol}}) \otimes_{\mathcal{O}_{\text{inf}}} (\mathcal{O}_{\text{inf}}, F_{\text{inf}}).$$

We say that $\mathcal{O}_{\text{inf}}^\tau$ is the τ -*edged localisation* of $\mathcal{O}_{\text{inf}}$. For an infinitesimal extension $(B, \underline{C})$ over A , write $B_{\text{inf}}^\tau \in \text{FilAlg}(A)$ for $\mathcal{O}_{\text{inf}}^\tau(B, \underline{C})$. For $e \geq 0$ we also write $\mathcal{I}_{\text{inf}}^{\tau, e}$ for $\text{Fil}_{\text{inf}}^e(\mathcal{O}_{\text{inf}}^\iota) \subseteq \mathcal{O}_{\text{inf}}^\iota$ with the filtration induced by the one of $\mathcal{O}_{\text{inf}}^\tau$.

Example 4.2.5. Let $\mathbb{F}_p[t]$ be the marked ring $(\mathbb{F}_p[t], t)$. The edge-type $\lambda : \mathbb{Z}_{>0} \rightarrow \mathbb{N}_\infty$ defined by $\lambda(e) = e$ corresponds to the overconvergent theory (see §6.2). In this case, for every $n, N > 0$, we have that

$$\mathcal{O}_{\text{inf}}^\iota(\mathbb{Z}/p^N[t], \mathbb{F}_p[t]) = \mathbb{Z}/p^N[t, t^{-1}]$$

and $\text{Fil}_{\lambda_n}^0 = \mathbb{Z}/p^N[t, \frac{p}{t^n}]$.

Construction 4.2.6. Let $\varphi : \mathbb{Z}_{>0} \rightarrow \mathbb{R}$ be a map of sets. We define $\tau^\varphi : \mathbb{Z}_{>0} \rightarrow \mathbb{N}_\infty$ inductively by

$$\tau^\varphi(e) := \max \left\{ \sup_{\varphi(u) \leq e} \{u\}, \max_{0 < f < e} \{\tau^\varphi(e - f) + \tau^\varphi(f)\} \right\}.$$

We say that τ^φ is the edge-type associated to the *decay-function* φ .

Definition 4.2.7. Let $\varphi_r : \mathbb{Z}_{>0} \rightarrow \mathbb{R}$ be the decay-function $\varphi_r(u) := \frac{\log_p(u)}{r}$, where r is a positive real number. We denote by θ^r the edge-type τ^{φ_r} . This type corresponds to the r -log-decay theory, [K-M16], [K-M21], [K-M22].

Example 4.2.8. When $r = 1$ we have that $\theta^1(e) = p^e$ and

$$\mathrm{Fil}_{\theta_n^1}^0 = \varinjlim_e \mathbb{Z}/p^N \left[t, \frac{p}{t^{np}}, \dots, \frac{p^e}{t^{np^e}} \right].$$

Example 4.2.9. Let $\underline{X}$ be the affine scheme $\mathrm{Spec}(C)$ over A with marking given by $f \in C$ nonzero-divisor. If $(B, \underline{C})$ is the marked infinitesimal extension with $B = C \oplus \Omega_{C/A}^1$ with the split square-zero C -algebra structure, we have

$$\mathrm{Fil}_{\tau}^0(\mathcal{O}_{\mathrm{inf}}^t)(B, \underline{C}) = C \oplus \frac{1}{f^{\tau(1)}} \Omega_{C/A}^1.$$

Construction 4.2.10. For an infinitesimal extension $(B, \underline{C})$ over A , an edge-type τ , and $i \geq 0$ we define

$$\Omega_{B/A}^{\tau, i} := \Omega_{B/A}^i \langle i \rangle \otimes_B B^{\tau}$$

where the tensor product is in $\mathrm{Fil}(\mathrm{Mod}(B))$. We say that the complex

$$B^{\tau} \xrightarrow{d} \Omega_{B/A}^{\tau, 1} \xrightarrow{d} \Omega_{B/A}^{\tau, 2} \rightarrow \dots$$

is the τ -edged localisation of the de Rham complex of B over A . Note that

$$\Omega_{B/A}^{\tau, i} = \bigwedge^i \Omega_{B/A}^{\tau, 1},$$

where the wedge product is in $\mathrm{FilMod}(B^{\tau})$. The constructions define filtered $\mathcal{O}_{\mathrm{inf}}^{\tau}$ -modules $\Omega_{\mathrm{inf}}^{\tau, i}$ over $(\underline{X}/A)_{\mathrm{inf}}$

Lemma 4.2.11. For every $(B, \underline{B/J}) \in \mathrm{INF}(\underline{X}/A)$ and every ideal $J \subseteq K \subseteq B$, the morphism

$$\mathcal{O}_{\mathrm{inf}}^{\tau}(B, \underline{B/J}) \rightarrow \mathcal{O}_{\mathrm{inf}}^{\tau}(B/K, \underline{B/J})$$

is a strict epimorphism of filtered B -modules.

Proof. By definition, it is enough to prove the statement both for $(\mathcal{O}_{\mathrm{inf}}, F_{\mathrm{inf}})$ and $(\mathcal{O}_{\mathrm{inf}}^t, \tau \star F_{\mathrm{pol}})$. The morphism $(B, \mathrm{Fil}_{\mathrm{inf}}) \rightarrow (B/K, \mathrm{Fil}_{\mathrm{inf}})$ is a strict epimorphism by construction. To prove that

$$(B^t, \tau \star F_{\mathrm{pol}}) \rightarrow ((B/K)^t, \tau \star F_{\mathrm{pol}})$$

is a strict epimorphism we can do it locally on $\mathrm{Spec}(B)$, so that we can assume $\underline{B/J}$ principal. We then deduce the result thanks to the observation that

$$\mathbb{M}_{\mathrm{inf}}(B, \underline{B/J}) \rightarrow \mathbb{M}_{\mathrm{inf}}(B/K, \underline{B/J})$$

is surjective. □

Construction 4.2.12. For $e \geq 0$ we write $\mathrm{INF}_e(\underline{X}/A) \subseteq \mathrm{INF}(\underline{X}/A)$ for the full subcategory of those $(B, \underline{B/J})$ with $J^e = 0$. There is a natural morphism of sites

$$\iota_e : \mathrm{INF}_e(\underline{X}/A) \rightarrow \mathrm{INF}(\underline{X}/A)$$

induced by the continuous functor $(B, \underline{B}/J) \mapsto (B/J^e, \underline{B}/J)$. We denote by $\mathcal{O}_{\text{inf},e}^\tau$ the filtered sheaf $\iota_{e*}(\mathcal{O}_{\text{inf}}^\tau)$ over $\text{INF}(\underline{X}/A)$ and $\mathcal{O}_{\text{inf},e}^\iota$ the sheaf $\iota_{e*}(\mathcal{O}_{\text{inf}}^\iota)$. Note that $\mathcal{O}_{\text{inf}}^\tau = \varprojlim_e \mathcal{O}_{\text{inf},e}^\tau$ since every $(B, \underline{B}/J) \in \text{INF}(\underline{X}/A)$ is in $\text{INF}_e(\underline{X}/A)$ for e big enough.

Lemma 4.2.13. *For every $e \geq 0$ have the following strict exact sequence of sheaves of filtered $\mathcal{O}_{\text{inf}}$ -modules*

$$0 \rightarrow \mathcal{I}_{\text{inf}}^{\tau,e} \rightarrow \mathcal{O}_{\text{inf}}^\tau \rightarrow \mathcal{O}_{\text{inf},e}^\tau \rightarrow 0.$$

Proof. The morphism $\mathcal{I}_{\text{inf}}^{\tau,e} \rightarrow \mathcal{O}_{\text{inf}}^\tau$ is a strict monomorphism from the definition, while $\mathcal{O}_{\text{inf}}^\tau \rightarrow \mathcal{O}_{\text{inf},e}^\tau$ is a strict epimorphism by Lemma 4.2.11. To prove the strict exactness in the middle it is enough to prove that the sequence of sheaves

$$0 \rightarrow \mathcal{I}_{\text{inf}}^e \mathcal{O}_{\text{inf}}^\iota \rightarrow \mathcal{O}_{\text{inf}}^\iota \rightarrow \mathcal{O}_{\text{inf},e}^\iota \rightarrow 0$$

obtained by forgetting the filtration is exact (in the middle). We can work Zariski locally on $\text{Spec}(B)$, so that we may assume $\underline{B}/J$ principal. If S is the multiplicative set generated by $\mathbb{M}_{\text{inf}}(B, \underline{B}/J)$ the result follows from the exactness of the sequence of B -modules.

$$0 \rightarrow J^e(S^{-1}B) \rightarrow S^{-1}B \rightarrow S^{-1}(B/J^e) \rightarrow 0.$$

□

4.3. Filtered divided power structures.

Definition 4.3.1. A *divided power ring* is a ring A endowed with an ideal I and a divided power structure γ on I . We denote such a datum by $(A, A/I, \gamma)$ and write $\text{Ring}^\sharp$ for the category they form.

Definition 4.3.2 (Filtered divided power structure). Let (A, Fil) be a filtered ring. A *filtered divided power structure* on (A, Fil) is a divided power structure on $\text{Fil}^1(A)$ such that $\gamma_n(\text{Fil}^1(A)) \subseteq \text{Fil}^n(A)$ for $n \geq 1$. We say that a filtered ring endowed with a filtered divided power structure is a *filtered divided power ring*. We write $\text{FilAlg}^\sharp(A, \text{Fil}, \gamma)$ for the category of filtered divided power rings over (A, Fil, γ) .

Proposition 4.3.3. *The forgetful functor*

$$\text{FilAlg}^\sharp(A, \text{Fil}, \gamma) \rightarrow \text{FilAlg}(A, \text{Fil})$$

admits a left adjoint D_γ .

Proof. For a filtered ring homomorphism $(A, \text{Fil}) \rightarrow (B, \text{Fil})$ we have to prove that there exists a filtered divided power ring $(D, \text{Fil}, \bar{\gamma})$ such that

$$\text{Hom}_{(A, \text{Fil}, \gamma)}((D, \text{Fil}, \bar{\gamma}), (C, \text{Fil}, \delta)) = \text{Hom}_{(A, \text{Fil})}((B, \text{Fil}), (C, \text{Fil}))$$

for $(C, \text{Fil}, \delta) \in \text{FilAlg}^\sharp(A, \text{Fil}, \gamma)$. □

Definition 4.3.4 (Filtered divided power envelope). For a filtered ring (B, Fil) over (A, Fil) we say that $D_\gamma(B, \text{Fil})$ is the *filtered divided power envelope* of (B, Fil) over (A, Fil, γ) .

Construction 4.3.5. For a set L and a filtered ring (A, Fil) , we denote by $(A\{x_\ell\}_{\ell \in L}, \text{Fil})$ the filtered ring with $A\{x_\ell\}_{\ell \in L}$ the polynomial divided power algebra over A freely generated by $\{x_\ell\}_{\ell \in L}$ endowed with the minimal filtration compatible with the one on A and such that $x_\ell^{[i]} \in \text{Fil}^i(A\{x_\ell\}_{\ell \in L})$ for $i \geq 1$.

If (A, Fil) is endowed with the structure of a filtered divided power structure γ , it induces a compatible filtered divided power structure δ on $A\{x_\ell\}_{\ell \in L}$ such that $\delta_n(x_\ell^{[m]}) = x_\ell^{[n+m]}$ for all $n, m \geq 0$. The filtered divided power algebra $(A\{x_\ell\}_{\ell \in L}, \text{Fil}, \delta)$ over (A, Fil, γ) is the *filtered polynomial divided power algebra over (A, Fil, γ) generated by $\{x_\ell\}_{\ell \in L}$* .

Construction 4.3.6. If (B, Fil, γ) is a filtered divided power ring over A , we have a filtered divided power de Rham complex $\Omega_{(B, \text{Fil}, \gamma)/A}^\bullet$ defined using divided power Kähler differentials. The filtration is the minimal one compatible with the one of B and such that $d(\text{Fil}^0(B)) \in \text{Fil}^1$.

Lemma 4.3.7 (Filtered Poincaré Lemma). *Let (B, Fil) be a filtered ring and let*

$$(P, \text{Fil}, \gamma) := (B\{x_\ell\}_{\ell \in L}, \text{Fil}, \gamma)$$

be a filtered polynomial divided power algebra over (B, Fil) . The complex of filtered (B, Fil) -modules

$$B \rightarrow P \rightarrow \Omega_{(P, \text{Fil}, \gamma)/B}^1 \rightarrow \Omega_{(P, \text{Fil}, \gamma)/B}^2 \rightarrow \dots$$

is homotopy equivalent to zero. In particular, for every filtered (B, Fil) -module M , we have that

$$M \rightarrow M \otimes_B P \rightarrow M \otimes_B \Omega_{(P, \text{Fil}, \gamma)/B}^1 \rightarrow M \otimes_B \Omega_{(P, \text{Fil}, \gamma)/B}^2 \rightarrow \dots$$

is strictly exact.

Proof. If $L = *$ we get the complex

$$(B, \text{Fil}) \rightarrow \bigoplus_{i=0}^{\infty} (B, \text{Fil}\langle -i \rangle) x^{[i]} \xrightarrow{d} \bigoplus_{i=1}^{\infty} (B, \text{Fil}\langle -i \rangle) x^{[i-1]} dx$$

which is homotopy equivalent to zero. For the general case just use that

$$\Omega_{(P, \text{Fil}, \gamma)/B}^\bullet = \bigwedge_{\ell \in L} \Omega_{(B\{x_\ell\}, \text{Fil}, \gamma)/B}^\bullet.$$

□

Lemma 4.3.8. *Let A be a ring, (B, Fil, γ) a divided power ring over A , and (P, Fil, γ) the divided power ring over (B, Fil, γ) generated by $\{x_\ell\}_{\ell \in L}$. For a filtered B -module M endowed with a flat connection*

$$M \rightarrow M \otimes_B \Omega_{(B, \text{Fil}, \gamma)/A}^1,$$

the map on de Rham complexes

$$M \otimes_B \Omega_{(B, \text{Fil}, \gamma)/A}^\bullet \rightarrow M \otimes_P \Omega_{(P, \text{Fil}, \gamma)/A}^\bullet$$

is a quasi-isomorphism.

4.4. Edged crystals. A filtered $\mathcal{O}_{\text{inf}}$ -module $\mathcal{F}$ over $\text{INF}(\underline{X}/A)$ induces for every $(B, \underline{C}) \in \text{INF}(\underline{X}/A)$ a filtered $\mathcal{O}$ -module $\mathcal{F}_B$ on the Zariski site of $\text{Spec}(B)$. We say that $\mathcal{F}$ is *locally quasi-coherent* if for every $(B, \underline{C}) \in \text{INF}(\underline{X}/A)$ the sheaf $\mathcal{F}_B$ is induced by some object in $\text{Fil}(\text{Mod}(B))$.

Construction 4.4.1. For a divided power ring $(A, A/I, \gamma)$, a marked scheme $\underline{X}$ over A/I , we define $(\mathcal{O}_{\text{cris}}, \text{Fil}_\gamma)$ to be the presheaf of filtered rings obtained by applying locally D_γ to $(\mathcal{O}_{\text{inf}}, \text{Fil}_{\text{inf}})$. For an edge-type τ , we also write $\mathcal{O}_{\text{cris}}^\tau = (\mathcal{O}_{\text{cris}}^\tau, \text{Fil}_\tau)$ for the presheaf of filtered rings over $\text{INF}(\underline{X}/A)$ obtained by applying D_γ to $\mathcal{O}_{\text{inf}}^\tau$. We also write $\mathcal{I}_{\text{cris}}^{\tau, e}$ for $\text{Fil}_\tau^e(\mathcal{O}_{\text{cris}}^\tau) \cdot \mathcal{O}_{\text{cris}}^\tau \subseteq \mathcal{O}_{\text{cris}}^\tau$ endowed with the filtration induced by Fil_τ by restriction. We say that $\mathcal{O}_{\text{cris}}^\tau$ is the τ -localisation of $\mathcal{O}_{\text{cris}}$. Similarly, we also write $\mathcal{O}_{\text{cris}}^{\tau_\infty}$ for the presheaf of negatively filtered rings

$$\tilde{\mathcal{O}}_{\text{cris}} \subseteq \text{Fil}_{\tau_1}^{-1}(\mathcal{O}_{\text{cris}}^\tau) \subseteq \text{Fil}_{\tau_2}^{-1}(\mathcal{O}_{\text{cris}}^\tau) \subseteq \dots$$

with $\tilde{\mathcal{O}}_{\text{cris}}$ at level 0.

Lemma 4.4.2. *For every τ , the presheaf $\mathcal{O}_{\text{cris}}^\tau$ is a locally quasi-coherent filtered $\mathcal{O}_{\text{inf}}$ -module.*

We have also some variants of τ -edged localisation.

Definition 4.4.3. The *hyperedged localisation* of $\mathcal{O}_{\text{cris}}$ is the sheaf of negatively filtered $\mathcal{O}_{\text{cris}}$ -algebras $(\mathcal{O}_{\text{cris}}^\tau, \text{Fil}_h)$, where Fil_h is the minimal filtration such that $\text{Fil}_h^0(\mathcal{O}_{\text{cris}}^\tau) = \tilde{\mathcal{O}}_{\text{cris}}$ and $\text{Fil}_h^{-1}(\mathcal{O}_{\text{cris}}^\tau)$ is locally generated by the inverses of the sections of $\mathbb{M}_{\text{inf}}$.

Definition 4.4.4. Let k be a perfect field, W its ring of Witt vectors, and K the fraction field of W . For a separated scheme X of finite type over k and an edge-type τ , the τ_∞ -edged crystalline cohomology complex of X is the complex

$$R\Gamma_{\tau_\infty\text{-cris}}(X/W) := \varinjlim_{X \subseteq Y} R\Gamma(\text{INF}(Y/W), \mathcal{O}_{\text{cris}}^{\tau_\infty}) \in D(\text{Ab}^{\mathbb{Z}})$$

where the colimit is over all $X \subseteq Y$ with Y proper. Similarly the *hyperedged crystalline cohomology complex* of X is the complex

$$R\Gamma_{h\text{-cris}}(X/W) := \varinjlim_{X \subseteq Y} R\Gamma(\text{INF}(Y/W), \mathcal{O}_{\text{cris}}^h) \in D(\text{Ab}^{\mathbb{Z}_{\leq 0}})$$

where again the colimit is over all $X \subseteq Y$ with Y proper. Note that when $\tau(1) \neq 0$, there is a natural morphism $\mathcal{O}_{\text{cris}}^h \rightarrow \mathcal{O}_{\text{cris}}^{\tau_\infty}$ which induces a morphism of complexes

$$R\Gamma_{h\text{-cris}}(X/W) \rightarrow R\Gamma_{\tau_\infty\text{-cris}}(X/W)$$

in $D(\text{Ab}^{\mathbb{Z}_{\leq 0}})$

Definition 4.4.5. A filtered $\mathcal{O}_{\text{cris}}^\tau$ -module is a module object over $\mathcal{O}_{\text{cris}}^\tau$ in $\text{Fil}(\text{Mod}(\mathcal{O}_{\text{inf}}))$. A τ -edged crystal over $\underline{X}$, is a locally quasi-coherent sheaf of $\mathcal{O}_{\text{cris}}^\tau$ -modules $\mathcal{F}$ over $\text{INF}(\underline{X}/A)$ such that for every morphism $(B_2, \underline{B_2/J_2}) \rightarrow (B_1, \underline{B_1/J_1})$ in $\text{INF}(\underline{X}/A)$, the comparison morphism

$$\mathcal{F}(B_1, \underline{B_1/J_1}) \otimes_{\mathcal{O}_{\text{cris}}^\tau(B_1, \underline{B_1/J_1})} \mathcal{O}_{\text{cris}}^\tau(B_2, \underline{B_2/J_2}) \rightarrow \mathcal{F}(B_2, \underline{B_2/J_2})$$

is an isomorphism of $\mathcal{O}_{\text{cris}}^\tau(B_2, \underline{B_2/J_2})$ -modules. We write $\text{C}^\tau(\underline{X}/A)$ for the category of τ -edged crystals over $\underline{X}$. We also say that a locally quasi-coherent filtered $\mathcal{O}_{\text{cris}}^\tau$ -module $\mathcal{F}$ is a τ -edged quasi-crystal if for every morphism $(B_2, \underline{B_2/J_2}) \rightarrow (B_1, \underline{B_1/J_1})$ in $\text{INF}(\underline{X}/A)$ with $B_1 \rightarrow B_2$ surjective

and $\underline{B_1/J_1} \rightarrow \underline{B_2/J_2}$ minimal, the comparison morphism

$$\mathcal{F}(B_1, \underline{B_1/J_1}) \otimes_{\mathcal{O}_{\text{cris}}^\tau(B_1, \underline{B_1/J_1})} \mathcal{O}_{\text{cris}}^\tau(B_2, \underline{B_2/J_2}) \rightarrow \mathcal{F}(B_2, \underline{B_2/J_2})$$

is surjective.

Construction 4.4.6. For a marked infinitesimal extension $(B, \underline{B/J})$ and $e \geq 0$, we write $(B_e\{1\}, \underline{B/J})$ for the product in $\text{INF}_e(\underline{X}/A)$ of two copies of $(B/J^e, \underline{B/J})$. We write $\Omega_{\text{inf}, e}^{\tau, 1}$ for the presheaf on $\text{INF}(\underline{X}/A)$ which sends $(B, \underline{B/J})$ to

$$\mathcal{I}_{\text{inf}}^{\tau, 1}(B_e\{1\}, \underline{B/J}) / \mathcal{I}_{\text{inf}}^{\tau, 2}(B_e\{1\}, \underline{B/J}).$$

Similarly, we define $\Omega_{\text{cris}, e}^{\tau, 1}$ as

$$\mathcal{I}_{\text{cris}}^{\tau, 1}(B_e\{1\}, \underline{B/J}) / \mathcal{I}_{\text{cris}}^{\tau, 2}(B_e\{1\}, \underline{B/J}).$$

Finally, we write $\Omega_{\text{inf}}^{\tau, 1}$ for $\varprojlim_e \Omega_{\text{inf}, e}^{\tau, 1}$ and $\Omega_{\text{cris}}^{\tau, 1}$ for $\varprojlim_e \Omega_{\text{cris}, e}^{\tau, 1}$.

Lemma 4.4.7. *Let $B \rightarrow C$ a flat epimorphism of rings, then $\mathbb{L}_{C/B} = 0$. In particular, for every ring homomorphism $A \rightarrow B$, we have $\Omega_{C/A}^1 = \Omega_{B/A}^1 \otimes_B C$.*

Proof. The first part is [Stacks, Tag 08R2]. By the base change triangle of the cotangent complex, we deduce that

$$\mathbb{L}_{C/A} = \mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} C.$$

After taking π_0 , this implies that $\Omega_{C/A}^1 = \Omega_{B/A}^1 \otimes_B C$. □

Lemma 4.4.8. *The filtered $\mathcal{O}_{\text{cris}}^\tau$ -modules $\Omega_{\text{cris}}^{\tau, i}$ are τ -edged quasi-crystals.*

Construction 4.4.9. We look at the analogue of the two universal thickenings of [Stacks, Tag 07KN] in our situation. We fix a marked infinitesimal extension $(B, \underline{B/J}) \in \text{INF}(\underline{X}/A)$. We write B_1 for the square-free split extension $B \oplus \Omega_{B/A}^1$. We get then a marked infinitesimal extension $(B_1, \underline{B/J}) \in \text{INF}(\underline{X}/A)$. We also write B_2 for the B -algebra

$$B \oplus \Omega_{B/A}^1 \oplus \Omega_{B/A}^1 \oplus \Omega_{B/A}^2$$

with algebra structure defined by

$$(f, \omega_1, \omega_2, \eta) \cdot (f', \omega'_1, \omega'_2, \eta') = (ff', f\omega'_1 + f'\omega_1, f\omega'_2 + f'\omega_2, f\eta' + f'\eta + \omega_1 \wedge \omega'_2 + \omega'_1 \wedge \omega_2).$$

Lemma 4.4.10. *There exist natural strict epimorphisms*

$$\mathcal{O}_{\text{cris}}^\tau(B_1, \underline{B/J}) \twoheadrightarrow \Omega_{\text{cris}}^{\tau, 0} \oplus \Omega_{\text{cris}}^{\tau, 1}$$

and

$$\mathcal{O}_{\text{cris}}^\tau(B_2, \underline{B/J}) \twoheadrightarrow \Omega_{\text{cris}}^{\tau, 0} \oplus \Omega_{\text{cris}}^{\tau, 1} \oplus \Omega_{\text{cris}}^{\tau, 1} \oplus \Omega_{\text{cris}}^{\tau, 2},$$

where we set

$$\Omega_{\text{cris}}^{\tau, i} := \Omega_{\text{cris}}^{\tau, i}(B, \underline{B/J}).$$

Proof. Since the claim is Zariski local, we may assume that the marking $\underline{B/J}$ is principal. Define $B^\iota := \mathcal{O}_{\text{inf}}^\iota(B)$ and consider the natural embedding $p_0: B \hookrightarrow B \oplus \Omega_{B/A}^1$ sending $b \mapsto (b, 0)$. Then one verifies that

$$(\mathcal{O}_{\text{inf}}^\iota(B_1, \underline{B/J}), \text{Fil}_{\text{pol}}) = (B^\iota, \text{Fil}_{\text{pol}}) \otimes_{B\langle 0 \rangle} B_1\langle 0 \rangle,$$

and similarly,

$$(B_1, \text{Fil}_{\text{inf}}) = (B, \text{Fil}_{\text{inf}}) \otimes_{B\langle 0 \rangle} (B\langle 0 \rangle \oplus \Omega_{B/A}^1\langle 1 \rangle).$$

It follows that

$$\mathcal{O}_{\text{inf}}^\tau(B_1, \underline{B/J}) = \left(\mathcal{O}_{\text{inf}}^\iota(B_1, \underline{B/J}), \tau \star \text{Fil}_{\text{pol}} \right) \otimes_{B_1\langle 0 \rangle} (B_1, \text{Fil}_{\text{inf}}) = (B^\iota, \text{Fil}_\tau) \otimes_{B\langle 0 \rangle} (B\langle 0 \rangle \oplus \Omega_{B/A}^1\langle 1 \rangle).$$

Thus, we obtain the decomposition

$$\mathcal{O}_{\text{inf}}^\tau(B_1, \underline{B/J}) = (B^\iota, \text{Fil}_\tau) \oplus \Omega_{\text{inf}}^{\tau,1}(B, \underline{B/J}).$$

Taking the filtered divided power envelope of $\mathcal{O}_{\text{inf}}^\tau(B_1, \underline{B/J})$ yields a strict epimorphism

$$\mathcal{O}_{\text{cris}}^\tau(B_1, \underline{B/J}) \twoheadrightarrow \Omega_{\text{cris}}^{\tau,0} \oplus \Omega_{\text{cris}}^{\tau,1},$$

where the kernel is generated by the divided powers $\gamma_n(\omega)$ for all $n \geq 2$ and $\omega \in \Omega_{B/A}^1$.

A similar argument shows that

$$\mathcal{O}_{\text{inf}}^\tau(B_2, \underline{B/J}) = (B^\iota, \text{Fil}_\tau) \oplus \Omega_{\text{inf}}^{\tau,1}(B, \underline{B/J}) \oplus \Omega_{\text{inf}}^{\tau,1}(B, \underline{B/J}) \oplus \Omega_{\text{inf}}^{\tau,2}(B, \underline{B/J}),$$

which implies the corresponding strict epimorphism for B_2 . $\square$

4.4.11. We are finally ready to define τ -edged and hyperedged crystalline cohomology

Definition 4.4.12. Let k be a perfect field, W its ring of Witt vectors, and K the fraction field of W . For a separated scheme X of finite type over k and an edge-type τ , the category of *coherent τ -edged isocrystals* over X , denoted by $\text{Isoc}^\tau(X/K)$, is the 2-colimit of the isogeny category of crystals in coherent $\mathcal{O}_{\text{cris}}^\tau$ -modules over $\text{INF}(\underline{Y}/W)$ for different embeddings $X \subseteq \underline{Y}$ with Y proper.

5. THE MARKED CONVERGENT SITE

5.1. Marked enlargements.

Definition 5.1.1. A p -adic ring is a p -complete ring. We write $\text{Ring}_p^\wedge \subseteq \text{Ring}$ for the full subcategory of p -adic rings. A morphism $A \rightarrow B$ is p -étale if for every $e \geq 0$ the quotient $A/p^e \rightarrow B/p^e$ is étale. For a p -adic ring A , we write $N_m(A) \subseteq \text{Jac}(A)$ for the ideal of elements $x \in A$ such that $x^{p^m} \in (p)$ and we write

$$N_\infty(A) := \sqrt{(p)} = \bigcup_{m=0}^{\infty} N_m(A).$$

Equivalently, $N_m(A)$ is the kernel of $A \rightarrow A/p \xrightarrow{F^m} A/p$. We denote by $A_{\bar{m}} \subseteq A/p$ the quotient $A/N_m(A)$ and by A_∞ the quotient $A/N_\infty(A)$. Note that by definition we have $A_{\bar{0}} = A/p$ and

$$A_\infty = A/\sqrt{(p)} = \varinjlim (A_{\bar{0}} \xrightarrow{F} A_{\bar{1}} \xrightarrow{F} A_{\bar{2}} \xrightarrow{F} \dots).$$

For an ideal $I \subseteq A$ containing p , we write $I^{(m)}$ for the ideal generated by p and the elements x^{p^m} with $x \in I$.

Definition 5.1.2. A *marked p -adic ring* is a p -adic ring A endowed with a marking on $A/N_\infty(A)$. We define $\mathbb{M}_p^\wedge := \mathrm{Spf}(\mathbb{Z}_p\langle t \rangle, t^{-1})$ and we denote by A^{mer} the localisation of A with respect to the local sections of $\mathbb{M}_p^\wedge$. The ring A^{mer} is endowed with a natural *polar filtration*. Let A be a p -adic ring and $\underline{X}$ a marked scheme over A/p . For $m \in \mathbb{Z}_{\geq 0}$, a *marked enlargement of level m* of $\underline{X}$ over A , is the datum of a p -torsionfree marked p -adic ring $\underline{B} \in \underline{\mathrm{Alg}}_A^\wedge$ endowed with a morphism $\mathrm{Spec}(\underline{B}_{\bar{m}}) \rightarrow \underline{X}$. We write $\mathrm{ENL}_m(\underline{X}/A)$ for the opposite of the category of marked enlargements of level m over A .

Definition 5.1.3. A family of morphisms $\{\underline{B}_k \rightarrow \underline{B}\}_{k \in K}$ in $\mathrm{ENL}_m(\underline{X}/A)$ is an *étale covering* if the following conditions are satisfied.

- (1) $N_m(B_k) = N_m(B)B_k$ for every k .
- (2) $\{\mathrm{Spec}(B_k) \rightarrow \mathrm{Spec}(B)\}_{k \in K}$ is a p -étale covering.
- (3) For every $k \in K$, the morphism $\mathrm{Spec}((\underline{B}_k)_\infty) \rightarrow \mathrm{Spec}(\underline{B}_\infty)$ is minimal.

The category $\mathrm{ENL}_m(\underline{X}/A)$ endowed with the p -étale topology is the *(big) marked convergent site*.

Definition 5.1.4. Let $\mathcal{O}_{\mathrm{conv}}$ for the sheaf of A -algebras over $\mathrm{ENL}_m(\underline{X}/A)$ defined by $\underline{B} \mapsto B$ and $\mathcal{N}_m \subseteq \mathcal{O}_{\mathrm{conv}}$ for the subsheaf $\underline{B} \mapsto N_m(B)$ for $m \in \mathbb{N}_\infty$.

5.2. Meromorphic functions. Let us construct the sheaf of *meromorphic functions*.

Construction 5.2.1. As in the infinitesimal case, for every marked enlargement $\underline{B}$, the topological spaces of $\mathrm{Spf}(B)$ and $\mathrm{Spec}(B_{\bar{m}})$ coincide. Therefore, there exists an open affine formal subscheme $\mathfrak{U} \subseteq \mathrm{Spf}(B)$ corresponding to the affine open $\mathrm{Spec}(B_{\bar{m}})^\iota \subseteq \mathrm{Spec}(B_{\bar{m}})$. We write B^ι for $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{U}})$ and we denote by $\mathcal{O}_{\mathrm{conv}}^\iota$ the sheaf over $\mathrm{ENL}_m(\underline{X}/A)$ given by $\underline{B} \mapsto B^\iota$. We define

$$\mathcal{O}_{\mathrm{conv}}^{\mathrm{mer}} \subseteq \mathcal{O}_{\mathrm{conv}}^\iota$$

to be the subsheaf of $\mathcal{O}_{\mathrm{conv}}$ -algebras locally generated by the inverses of the sections of $\mathbb{M}_{\mathrm{conv}}$. This is called the sheaf of *meromorphic functions*. It is endowed with the *polar filtration* defined by

$$\mathrm{Fil}_{\mathrm{pol}}^0(\mathcal{O}_{\mathrm{conv}}^{\mathrm{mer}}) := \mathrm{im}(\mathcal{O}_{\mathrm{conv}} \rightarrow \mathcal{O}_{\mathrm{conv}}^\iota)$$

and $\mathrm{Fil}_{\mathrm{pol}}^{-1}(\mathcal{O}_{\mathrm{conv}}^{\mathrm{mer}})$ locally generated as a $\mathcal{O}_{\mathrm{conv}}$ -module by the inverses of the sections of $\mathbb{M}_{\mathrm{conv}}$.

Example 5.2.2. Let $\mathbb{F}_p[t]$ be the marked ring $(\mathbb{F}_p[t], t^{-1})$. We write $\mathbb{Z}_p\langle t \rangle_t^{\mathrm{mer}}$ for $\mathcal{O}_{\mathrm{conv}}^{\mathrm{mer}}(\mathbb{Z}_p\langle t \rangle, \mathbb{F}_p[t])$ endowed with its polar negative filtration. We have

$$\mathbb{Z}_p\langle t \rangle_t^{\mathrm{mer}} = S^{-1}\mathbb{Z}\langle t \rangle$$

where S is the multiplicative set of series congruent to t^n for some $n \geq 0$. Equivalently, by Weierstrass preparation theorem, we have

$$S^{-1}\mathbb{Z}_p\langle t \rangle = S_{\mathrm{alg}}^{-1}\mathbb{Z}_p\langle t \rangle,$$

where $S_{\mathrm{alg}} \subseteq S$ is the subset of polynomials. These meromorphic functions form a subalgebra of the *(relatively) overconvergent functions in 0*, denoted by $\mathbb{Z}_p\langle t \rangle_t^\dagger \subseteq \mathbb{Z}_p\langle t, t^{-1} \rangle$, i.e., functions converging in the annulus $1 - \epsilon \leq |t|_p \leq 1$ for some $\epsilon > 0$. The polar filtration $\mathrm{Fil}_{\mathrm{pol}}^{-n}$ corresponds to taking the $\mathbb{Z}_p\langle t \rangle$ -submodule of $\mathbb{Z}_p\langle t \rangle_t^{\mathrm{mer}}$ generated by the inverses of the series congruent to t^n modulo p .

Notation 5.2.3. For a p -adic ring A and $n \geq 0$, write $\text{Fil}_{\lambda_n}^{-1}(A\langle t \rangle, t^{-1})$ for the $A\langle t \rangle$ -submodule of $A\langle t, t^{-1} \rangle$ generated by the series of the form

$$\frac{1}{t^n} \sum_{j=0}^{\infty} b_j \frac{p^j}{t^{jn}}$$

with $b_j \in A$ (without convergence assumptions).

Proposition 5.2.4. *Let A be a p -adic ring such that A/p contains an infinite field k , then*

$$\text{Fil}_{\lambda_n}^{-1}(A\langle t \rangle, t^{-1}) = \text{Fil}_{\text{pol}}^{-n}(A\langle t \rangle_t^{\text{mer}})^{\text{cl}},$$

where $(-)^{\text{cl}}$ denotes the p -adic closure in $A\langle t, t^{-1} \rangle$. In particular,

$$A\langle t \rangle_t^{\dagger} = \bigcup_{n=0}^{\infty} \text{Fil}_{\text{pol}}^{-n}(A\langle t \rangle_t^{\text{mer}})^{\text{cl}}.$$

Proof. Set $B_n := \text{Fil}_{\text{pol}}^{-n}(A\langle t \rangle_t^{\text{mer}})$ and $\tilde{B} := \bigcup_{n=0}^{\infty} B_n^{\text{cl}}$, thought as $A\langle t \rangle$ -submodules of $A\langle t, t^{-1} \rangle$. By construction, $\tilde{B}$ is a $A\langle t \rangle$ -subalgebra of $A\langle t \rangle_t^{\dagger}$. We have to prove that for every $n \geq 0$ and every sequence $(b_j)_{j \geq 0}$ in A , then

$$f := \frac{1}{t^n} \sum_{j=0}^{\infty} b_j \frac{p^j}{t^{jn}} \in B_n^{\text{cl}}.$$

Choose a sequence $(c_i)_{i \geq 0} \in A$ with distinct reductions in $k \subseteq A/p$. For $m \geq 0$, solve the Vandermonde system

$$\sum_{i=0}^m a_i c_i^j = b_j \quad \text{for } 0 \leq j \leq m,$$

where $a_i \in A$. Note that the Vandermonde matrix $(c_i^j)_{0 \leq i, j \leq m}$ is invertible because the reduction modulo p of the determinant lies in k^* . If we write

$$f_m := \sum_{i=0}^m \frac{a_i}{t^n - c_i p} \in A,$$

using the geometric expansion

$$\frac{a_i}{t^n - c_i p} = \frac{a_i}{t^n} \left(1 - \frac{c_i p}{t^n}\right)^{-1} = \frac{a_i}{t^n} \sum_{j=0}^{\infty} \left(\frac{c_i p}{t^n}\right)^j$$

we can write

$$f_m = \sum_{j=0}^{\infty} \sum_{i=0}^m a_i c_i^j \frac{p^j}{t^{(j+1)n}} = \frac{1}{t^n} \sum_{j=0}^m b_j \frac{p^j}{t^{jn}} + \sum_{j=m+1}^{\infty} \sum_{i=0}^m a_i c_i^j \frac{p^j}{t^{(j+1)n}}.$$

This shows that f_m have the same reduction as f in $(A/p^{m+1})[t, t^{-1}]$, and this proves the desired result. $\square$

6. COMPARISON THEOREMS

6.1. Computing edged crystalline cohomology.

Lemma 6.1.1. *If $\mathcal{M}$ is a locally quasi-coherent sheaf of $\mathcal{O}_{\text{inf}}$ -modules over $\text{INF}(\underline{X}/A)$, then for every $(B, \underline{C}) \in \text{INF}(\underline{X}/A)$ we have*

$$H^i((B, \underline{C}), \mathcal{M}) = 0$$

for $i > 0$.

Proof. This can be proven using Serre vanishing as in [Stacks, Tag 07JJ]. $\square$

Corollary 6.1.2. *If we denote by $\mathcal{C}_e$ the category $\text{INF}_e(\underline{X}/A)$ endowed with the chaotic topology, then for every locally quasi-coherent sheaf of $\mathcal{O}_{\text{inf}}$ -modules $\mathcal{M}$ over $\text{INF}_e(\underline{X}/A)$, we have*

$$R\Gamma(\text{INF}_e(\underline{X}/A), \mathcal{M}) = R\Gamma(\mathcal{C}_e, \mathcal{M}).$$

Proof. The identity $\mathcal{C}_e \rightarrow \text{INF}_e(\underline{X}/A)$ induces a morphism of ringed sites

$$f: (\text{INF}_e(\underline{X}/A), \mathcal{O}_{\text{inf}}) \rightarrow (\mathcal{C}_e, \mathcal{O}_{\text{inf}}).$$

By Lemma 6.1.1, for a quasi-coherent sheaf of $\mathcal{O}_{\text{inf}}$ -modules $\mathcal{M}$, the higher direct images $R^i f_* \mathcal{M}$ vanish for $i > 0$. The result then follows from the observation that $f_* \mathcal{M} = \mathcal{M}$. See [Stacks, Tag 07JK] for more details. $\square$

Construction 6.1.3. Let $\underline{X}$ be an affine marked scheme over A of the form $\text{Spec}(\underline{C})$ and let e be a positive integer. Choose a polynomial A -algebra P which admits a surjection $P \twoheadrightarrow C$. We define $J := \ker(P \rightarrow C)$ and $Q_e := P/J^e$ for $e \geq 1$. We denote by $(Q_e\{*\}, \underline{C})$ the Čech nerve of $(Q_e, \underline{C})$ in $\text{INF}_e(\underline{X}/A)$. For every $i, n \geq 0$ we write $\Omega_e^{\tau, i}(n)$ for $\Omega_{\text{cris}}^{\tau, i}(Q_e(n), \underline{C})$. The A -modules $\Omega_e^{\tau, 0}(n) = \mathcal{O}_{\text{cris}}^{\tau}(Q_e\{*\}, \underline{C})$ are divided power rings over (A, Fil, γ) that we also denote by $D_e^{\tau}(n)$. When $n = 0$ we often drop the index and we write, for example,

$$D_e^{\tau} \rightarrow \Omega_e^{\tau, 1} \rightarrow \Omega_e^{\tau, 2} \rightarrow \dots$$

for the τ -edged crystalline de Rham complex of $(Q_e, \underline{C})$.

Lemma 6.1.4. *The marked infinitesimal extension $(Q_e, \underline{C})$ is a weakly final object of $\text{INF}_e(\underline{X}/A)$.*

Proof. Let $(B, \underline{C}')$ be another object in $\text{INF}_e(\underline{X}/A)$. We choose a morphism $P \rightarrow B$ lifting the composition $P \rightarrow C \rightarrow C'$. If K is the kernel of $B \rightarrow C'$, by assumption $K^e = 0$, thus $P \rightarrow B$ factors through $Q_e \rightarrow B$. By construction, we get a commutative diagram,

$$\begin{array}{ccc} Q_e & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & C', \end{array}$$

as we wanted. $\square$

Lemma 6.1.5. *For every cosimplicial filtered module $M_e\{*\}$ over $D_e^{\tau}\{*\}$, the cosimplicial filtered module*

$$M_e\{0\} \otimes_{D_e^{\tau}\{0\}} \Omega_e^{\tau, i}\{0\} \rightarrow M_e\{1\} \otimes_{D_e^{\tau}\{1\}} \Omega_e^{\tau, i}\{1\} \rightarrow \dots$$

is homotopic to zero for every $i \geq 1$.

Proof. Since

$$\Omega_e^{\tau,1}(n) = \Omega_{\text{cris}}^1(Q_e(n), C)\langle 1 \rangle \otimes_{D_e(n)} D_e^\tau(n)$$

we deduce that $\Omega_e^{\tau,1}\{*\}$ is obtained from $\Omega_{\text{cris}}^1(Q_e\{*\}, C)\langle 1 \rangle$ by base change with respect to the morphism of cosimplicial filtered rings $D_e\{*\} \rightarrow D_e^\tau\{*\}$. Thanks to [Stacks, Tag 07L9] the complex $\Omega_{\text{cris}}^1(Q_e\{*\}, C)$ is homotopic to zero as a $D_e\{*\}$ -module. The result then follows from [Stacks, Tag 07KQ]. $\square$

Construction 6.1.6. Let $\mathcal{M}$ be a sheaf of filtered $\mathcal{O}_{\text{inf}}^\tau$ -modules over $\text{INF}(\underline{X}/A)$. We denote by $M_e\{*\}$ the cosimplicial module $\mathcal{M}(Q_e\{*\}, \underline{C})$. We also write M_e for $M_e\{0\}$. If $\mathcal{M}$ is a τ -edged crystal the two projections $\text{pr}_i : (Q_e\{1\}, \underline{C}) \rightarrow (Q_e, \underline{C})$ with $i = 1, 2$ induce isomorphisms

$$\text{pr}_1^*(M_e) \xrightarrow{\sim} M_e\{1\} \xrightarrow{\sim} \text{pr}_2^*(M_e),$$

which define a filtered flat connection

$$M_e \rightarrow M_e \otimes_{D_e^\tau} \Omega_e^{\tau,1}$$

with de Rham complex $M_e \otimes_{D_e^\tau} \Omega_e^{\tau,\bullet}$.

Proposition 6.1.7. *If $\mathcal{M}$ is a τ -edged quasi-crystal of $\text{INF}(\underline{X}/A)$, then there exists a quasi-isomorphism*

$$R\Gamma(\text{INF}_e(\underline{X}/A), \mathcal{M}) \xrightarrow{\sim} (M_e\{0\} \rightarrow M_e\{1\} \rightarrow M_e\{2\} \rightarrow \dots).$$

Proof. The result follows from Corollary 6.1.2 and Lemma 6.1.4 as in [Stacks, Tag 07JN]. $\square$

Corollary 6.1.8. *If $\mathcal{M}$ is a τ -edged quasi-crystal, then*

$$H^j(\text{INF}_e(\underline{X}/A), \mathcal{M} \otimes_{\mathcal{O}_{\text{cris}}^\tau} \Omega_{\text{cris}}^{\tau,i}) = 0$$

for all $i > 0$ and $j \geq 0$.

Proof. By Lemma 4.4.8, the sheaves $\Omega_{\text{cris}}^{\tau,i}$ are τ -edged quasi-crystals, which implies that $\mathcal{M} \otimes_{\mathcal{O}_{\text{cris}}^\tau} \Omega_{\text{cris}}^{\tau,i}$ is a τ -edged quasi-crystal. Thanks to Proposition 6.1.7 we can then compute its cohomology using the complex $M_e\{*\} \otimes_{D_e^\tau\{*\}} \Omega_e^{\tau,i}\{*\}$, which is acyclic by Lemma 6.1.5. $\square$

Lemma 6.1.9 (Bhatt–de Jong). *Let $\mathcal{A}$ be an abelian category and let*

$$K^\bullet\{0\} \rightrightarrows K^\bullet\{1\} \rightrightarrows \dots$$

be a cosimplicial cochain complex of $\mathcal{A}$ such that for every $b \geq 0$ the complex $K^\bullet(b)$ is concentrated in non-negative degrees. For $0 \leq i \leq b$ write $\alpha_{i,b} : K^\bullet\{0\} \rightarrow K^\bullet(b)$ for the morphism of complexes induced by the morphism $[0] \rightarrow [0, \dots, b]$ which sends 0 to i . Suppose that the following conditions are satisfied.

- (1) *For every $0 \leq i \leq b$, the morphism $\alpha_{i,b}$ is a quasi-isomorphism.*
- (2) *For every $a > 0$, the cochain complex associated to $K^a\{*\}$ is acyclic.*

If $K^{\bullet,}$ is the double complex associated to $K^\bullet\{*\}$, then both $K^{0,*}$ and $K^{\bullet,0}$ are quasi-isomorphic to $\text{Tot}(K^{\bullet,*})$.*

Proof. We first note that for every $0 \leq i \leq b$ and $j \geq 0$, the isomorphisms

$$\alpha_{i,b}: H^j(K^\bullet\{0\}) \xrightarrow{\sim} H^j(K^\bullet\{b\})$$

have as common inverse the morphism

$$H^j(K^\bullet\{b\}) \rightarrow H^j(K^\bullet\{0\})$$

induced by $[0, \dots, b] \rightarrow [0]$. In particular, they are independent of i . When taking the first spectral sequence associated to the double complex $K^{\bullet,*}$, the differentials of the first page are

$$H^j(K^{0,*}) \xrightarrow{0} H^j(K^{1,*}) \xrightarrow{\sim} H^j(K^{2,*}) \xrightarrow{0} H^j(K^{3,*}) \rightarrow \dots$$

This implies that $K^{0,*}$ is quasi-isomorphic to $\text{Tot}(K^{\bullet,*})$. Looking at the second spectral sequence, Condition (2) implies that $K^{\bullet,0}$ is also quasi-isomorphic to $\text{Tot}(K^{\bullet,*})$. $\square$

Theorem 6.1.10. *Let $\mathcal{M}$ be a τ -edged crystal over $\text{INF}(\underline{X}/A)$. There exists a compatible system of quasi-isomorphisms of filtered complexes*

$$R\Gamma(\text{INF}_e(\underline{X}/A), \mathcal{M}) \xrightarrow{\sim} M_e \otimes_{D_e^\tau} \Omega_e^{\tau,\bullet}$$

indexed by $e \geq 1$.

Proof. We want to apply Lemma 6.1.9. We consider the double complex $K^{\bullet,*}$ of filtered modules defined by

$$K^{a,b} := M_e \otimes_{D_e^\tau} \Omega_e^{\tau,a}\{b\}.$$

By Lemma 6.1.5, the columns $K^{a,*}$ are acyclic when $a > 0$ and $K^{0,*}$ is quasi-isomorphic to $R\Gamma(\text{INF}_e(\underline{X}/A), \mathcal{M})$ thanks to Proposition 6.1.7. By Lemma 4.3.8, we deduce that for every $0 \leq i \leq b$ the morphism

$$\alpha_{i,b}: M_e \otimes_{D_e^\tau} \Omega_e^{\tau,\bullet}\{0\} \rightarrow M_e \otimes_{D_e^\tau} \Omega_e^{\tau,\bullet}\{b\}$$

is a quasi-isomorphism. $\square$

6.2. Comparison with rigid cohomology.

6.2.1. We write (A_n, Fil) for the projective limit

$$\varprojlim_e \mathcal{O}_{\text{inf}}^{\lambda_n}(\mathbb{Z}/p^e[t], (\mathbb{F}_p[t], t)).$$

By definition, $\text{Fil}^0(A_n)$ is the projective limit

$$\varprojlim_e \mathbb{Z}[t, \frac{p}{t^n}]/(\frac{p}{t^n})^e \subseteq \varprojlim_e \mathbb{Z}/p^e[t, t^{-1}] = \mathbb{Z}_p\langle t, t^{-1} \rangle.$$

We denote by $\mathbb{Z}_p\langle t, t^{-1} \rangle^-$ the $\mathbb{Z}_p$ -submodule of $\mathbb{Z}_p\langle t, t^{-1} \rangle$ of series of the form $\sum_{i=1}^{\infty} a_i t^{-i}$ with $a_i \in \mathbb{Z}_p$ and by $\text{Fil}^0(A_n)^-$ the intersection $\text{Fil}^0(A_n) \cap \mathbb{Z}_p\langle t, t^{-1} \rangle^-$.

Lemma 6.2.2. *Every element of $\text{Fil}^0(A_n)^-$ can be written uniquely in the form*

$$\sum_{i=1}^{\infty} b_i(t) \left(\frac{p}{t^n}\right)^i$$

where each $b_i(t) \in \mathbb{Z}_p[t]$ is a polynomial of degree at most $n-1$.

Proof. This follows from the analogous result modulo $(\frac{p}{t^n})^e$ for every e . $\square$

Write $B_{n,\mathbb{Q}_p}$ for the (unfiltered) $\mathbb{Q}_p$ -algebra $\mathrm{Fil}^0(A_n)[\frac{1}{p}] = A_n[\frac{1}{p}]$ and $C_{n,\mathbb{Q}_p}$ for the subalgebra of $\mathbb{Z}_p\langle t, t^{-1} \rangle[\frac{1}{p}]$ of series $\sum_{j=-\infty}^{\infty} a_j t^j$ such that $v_p(a_j) \geq \lceil -\frac{j}{n} \rceil$ for j small enough.

Lemma 6.2.3. *The algebra $C_{n,\mathbb{Q}_p}$ coincides with the subset of $f \in \mathbb{Z}_p\langle t, t^{-1} \rangle[\frac{1}{p}]$ which can be written as a sum $f = f_+ + f_-$ with $f_+ \in \mathbb{Z}_p\langle t \rangle[\frac{1}{p}]$ and $f_- \in \mathrm{Fil}^0(A_n)^-[\frac{1}{p}]$. In particular, $B_{n,\mathbb{Q}_p} = C_{n,\mathbb{Q}_p}$.*

Proof. It is enough to prove the result for those $f = \sum_{j=-\infty}^{\infty} a_j t^j \in C_{n,\mathbb{Q}_p}$ with $a_j = 0$ for $j \geq 0$. Write e_k for $\lceil \frac{k}{n} \rceil$ and suppose that $v_p(a_{-k}) \geq e_k$ for $m \gg 0$ and $k > mn$, then

$$f = \sum_{k=1}^{mn} a_{-k} t^{-k} + \sum_{k=mn+1}^{\infty} p^{e_k} a'_{-k} t^{-k}$$

where $a'_{-k} = \frac{a_{-k}}{p^{e_k}} \in \mathbb{Z}_p$. Since

$$\sum_{k=mn+1}^{\infty} p^{e_k} a'_{-k} t^{-k} = \sum_{i=m+1}^{\infty} \sum_{k=(i-1)n+1}^{in} p^i a'_{-k} t^{in-k} t^{-in} = \sum_{i=m+1}^{\infty} b_i(t) \left(\frac{p}{t^n}\right)^i$$

with

$$b_i(t) := \sum_{k=(i-1)n+1}^{in} a'_{-k} t^{in-k} \in \mathbb{Z}_p[t]$$

of degree at most $n-1$, we conclude by Lemma 6.2.2. $\square$

Corollary 6.2.4. *There exists a natural isomorphism*

$$\varinjlim_n B_{n,\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{Q}_p\langle t \rangle_t^\dagger,$$

where $\mathbb{Q}_p\langle t \rangle_t^\dagger \subseteq \mathbb{Q}_p\langle t, t^{-1} \rangle$ is the subring of series which are overconvergent at 0.

7. A STACKY APPROACH

7.1. The convergent stack. We reinterpret the convergent site construction by associating to schemes over $\mathbb{F}_p$ a *convergent stack*.

Definition 7.1.1. Let $\mathrm{Aff}_{\mathbb{Q}_p}^\wedge$ denote the category of complete affinoid rings (R, R^+) over $(\mathbb{Q}_p, \mathbb{Z}_p)$, that we call *p-adic affinoid rings*. For a scheme X over $\mathbb{F}_p$ and $m \in \mathbb{Z}_{\geq 0}$, we define the functor

$$X_{\mathrm{conv}}^{(m)} : \mathrm{Aff}_{\mathbb{Q}_p}^\wedge \rightarrow \mathrm{Set}, \quad (R, R^+) \mapsto X(R^+/N_m(R^+)).$$

The *convergent stack* is the functor

$$X_{\mathrm{conv}} := \varinjlim_m X_{\mathrm{conv}}^{(m)}.$$

Let us compute this functor for semi-perfect schemes.

Construction 7.1.2. Let C be a semi-perfect ring and write J for $\ker(W(C^\flat) \rightarrow C^\flat \rightarrow C)$. For every $m \in \mathbb{Z}_{\geq 0}$, we define

$$\tilde{A}_{\mathrm{conv}}^{(m)}(C) := \mathrm{Bl}_{J^{(m)}}(W(C^\flat))$$

endowed with the standard $\mathbb{Z}_{\geq 0}$ -grading. The p -completed graded localisation of $\tilde{A}_{\text{conv}}^{(m)}(C)$ with respect to $p\langle 1 \rangle$ is the p -adic algebra

$$A_{\text{conv}}^{(m)}(C) := W(C^\flat)[\frac{J^{(m)}}{p}]^\wedge.$$

We also consider the ramified variant

$$\tilde{A}_{\text{conv}}^{(m,n)}(C) := \text{Bl}_{J^{(m,n)}}^{1/p^n} \left(W(C^\flat)[p^{1/p^n}] \right),$$

where $\text{Bl}_-^{1/p^n}(-)$ is the $\frac{1}{p^n}\mathbb{Z}_{\geq 0}$ -graded blow up algebra and $J^{(m,n)} := (p^{1/p^n}, J^{(m-n)})$. There are natural morphisms

$$\tilde{A}_{\text{conv}}^{(m,0)}(C) \rightarrow \tilde{A}_{\text{conv}}^{(m,1)}(C) \rightarrow \tilde{A}_{\text{conv}}^{(m,2)}(C) \rightarrow \dots$$

We denote by $\tilde{B}_{\text{conv}}^{(m)}(C), B_{\text{conv}}^{(m)}(C), \dots$ the extensions to $\mathbb{Q}_p$. Let $\tilde{X}_{\text{conv}}^{(m,n)}$ be the generic fibre of the p -adic completion of

$$\text{Proj}(\tilde{B}_{\text{conv}}^{(m,n)}(C), \tilde{A}_{\text{conv}}^{(m,n)}(C)).$$

We get a tower

$$(7.1) \quad \dots \rightarrow \tilde{X}_{\text{conv}}^{(m,2)} \rightarrow \tilde{X}_{\text{conv}}^{(m,1)} \rightarrow \tilde{X}_{\text{conv}}^{(m,0)}.$$

Lemma 7.1.3. *The projective limit $\tilde{X}_{\text{conv}}^{(m,\infty)}$ of (7.1) is a perfectoid space.*

Lemma 7.1.4. *For a semi-perfect ring, the adic space $\text{Spa}(B_{\text{conv}}^{(m)}(C), A_{\text{conv}}^{(m)}(C))$ is preperfectoid, hence sheafy.*

Proof. Set $A := A_{\text{conv}}^{(m)}(C)$ and $B := B_{\text{conv}}^{(m)}(C)$. Consider the perfectoid affinoid ring $(R, R^+) \in \text{Aff}_{\mathbb{Q}_p}^\wedge$ obtained as the p -adic completion of $(\mathbb{Q}_p[p^{1/p^\infty}], \mathbb{Z}_p[p^{1/p^\infty}])$. Form the tensor product in $\text{Aff}_{\mathbb{Q}_p}^\wedge$

$$(B', A') := (B, A) \otimes_{(\mathbb{Q}_p, \mathbb{Z}_p)} (R, R^+).$$

We have that $\text{Spa}(B', A')$ is an affinoid open of $\tilde{X}_{\text{conv}}^{(m,\infty)}$, hence perfectoid by Lemma 7.1.3. $\square$

Lemma 7.1.5. *For a semi-perfect ring C , the adic space $\text{Spa}(B_{\text{conv}}^{(m)}(C), A_{\text{conv}}^{(m)}(C))$ represents the functor $(\text{Spec}(C))_{\text{conv}}^{(m)}$.*

Proof. For $(R, R^+) \in \text{Aff}_{\mathbb{Q}_p}^\wedge$, a morphism

$$(B_{\text{conv}}^{(m)}(C), A_{\text{conv}}^{(m)}(C)) \rightarrow (R, R^+)$$

corresponds to a continuous homomorphism $f : W(C^\flat) \rightarrow R^+$ such that $f(J) \subseteq N_m(R^+)$. Hence it is equivalent to the datum of a morphism $C \rightarrow R^+/N_m(R^+)$, as we wanted. $\square$

Proposition 7.1.6. *The assignment $X \mapsto X_{\text{conv}}$ defines a functor*

$$\{\text{Semi-perfect schemes}/\mathbb{F}_p\} \rightarrow \{\text{Preperfectoid spaces}/\mathbb{Q}_p\}.$$

Proof. For semi-perfect affines this follows from Lemma 7.1.4 and Lemma 7.1.5. $\square$

7.2. The marked convergent stack.

Definition 7.2.1. A *marked p -adic affinoid ring* $(R, \underline{R}^+)$ is a p -adic affinoid ring (R, R^+) endowed with a marking on $R^+/N_\infty(R^+)$. We denote by $\underline{\text{Aff}}_{\mathbb{Q}_p}^\wedge$ the category of p -adic marked affinoid rings. For a marked scheme $\underline{X}$ over $\mathbb{F}_p$ and $m \in \mathbb{Z}_{\geq 0}$, we define the functor

$$\underline{X}_{\text{conv}}^{(m)} : \underline{\text{Aff}}_{\mathbb{Q}_p}^\wedge \rightarrow \text{Set}, \quad (R, \underline{R}^+) \mapsto \underline{X}(\underline{R}_m^+)$$

and we take

$$\underline{X}_{\text{conv}} := \varinjlim_m \underline{X}_{\text{conv}}^{(m)}.$$

We say that $\underline{X}_{\text{conv}}$ is the *marked convergent stack* of $\underline{X}$.

Definition 7.2.2. A *meromorphic coherent module* over $\underline{X}_{\text{conv}}$ is the datum of a coherent module $M_{(A, \underline{A}^+)} \in \text{Coh}(A^{\text{mer}})$ for every morphism $\text{Spa}(A, \underline{A}^+) \rightarrow \underline{X}_{\text{conv}}$ with $(A, \underline{A}^+) \in \underline{\text{Aff}}_{\mathbb{Q}_p}^\wedge$, together with the datum of isomorphisms

$$M_{(A, \underline{A}^+)} \otimes_{A^{\text{mer}}} B^{\text{mer}} \xrightarrow{\sim} M_{(B, \underline{B}^+)}$$

for every commutative triangle

$$\begin{array}{ccc} \text{Spa}(B, \underline{B}^+) & \xrightarrow{\quad} & \text{Spa}(A, \underline{A}^+) \\ & \searrow & \swarrow \\ & \underline{X}_{\text{conv}} & \end{array}$$

We write $\text{Coh}^{\text{mer}}(\underline{X}_{\text{conv}})$ for the category they form.

Similarly, a τ_∞ -*edged coherent module* is the datum of a coherent module $M_{(A, \underline{A}^+)} \in \text{Coh}(A^{\tau_\infty})$ for every morphism $\text{Spa}(A, \underline{A}^+) \rightarrow \underline{X}_{\text{conv}}$, additionally endowed with analogous compatibility isomorphisms. We write $\text{Coh}^{\tau_\infty}(\underline{X}_{\text{conv}})$ for the category of τ_∞ -edged coherent modules.

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