

CRYSTALS AND PARABOLICITY CONJECTURE — COURSE OUTLINE

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As a consequence of his theory on variations of *Hodge structure*, Deligne proved the following fundamental result.

Theorem 1 (Deligne’s semi-simplicity theorem). *For any smooth projective morphism $f: X \rightarrow S$ of smooth algebraic complex varieties, the local systems $R^i f_* \mathbb{Q}$ are semi-simple.*

He also proved an analogous result in positive characteristic: over $\overline{\mathbb{F}}_p$ the lisse \mathbb{Q}_ℓ -sheaves $R^i f_* \mathbb{Q}_\ell$ are semi-simple for $\ell \neq p$. In this setting, Hodge theory is replaced by the *theory of weights* for ℓ -adic cohomology. These theorems show that the associated *monodromy groups* are reductive and play a central role in arithmetic geometry.

In the crystalline setting, however, the situation differs. The higher direct images $R^i f_* \mathcal{O}_{\text{cris}}$ are not semi-simple as isocrystals. The obstruction is well understood: there is an additional filtration, the *slope filtration*, which is preserved by the p -adic connection. Thanks to the *parabolicity conjecture* (now a theorem), this filtration is in fact the *only* obstruction to semi-simplicity; namely, the associated graded object is a semi-simple isocrystal [D’A23]. In particular, the lisse \mathbb{Q}_p -sheaves $R^i f_* \mathbb{Q}_p$ are semi-simple.

During these lectures, we will introduce the theory of crystals as a tool for studying the *p -divisible groups* of abelian varieties and their deformations. We will cover fundamental results, such as the existence of the *slope filtration* and the construction of the *crystalline Dieudonné module functor*. We will also introduce *overconvergent isocrystals*, their *Tannakian formalism*, and the *independence-of- ℓ phenomenon*: for an irreducible overconvergent F -isocrystal over a finite field, there exist ℓ -adic companions whose monodromy groups are “isomorphic” [D’A20]. In the end, we will state the parabolicity conjecture, sketch some ideas of its proof, and explore some of the applications as the proof of the *Hecke orbit conjecture* [DvH22].

The course will assume familiarity with basic algebraic geometry, the étale/fppf/fpqc topologies, and some exposure to algebraic groups. Prior knowledge of Witt vectors, while not essential, will be beneficial. During the lectures, we will explain concrete examples and the geometric intuition behind technical definitions. For further reading, the main references are [Ked22] and [D’A24].

PROJECTS

Project A: Isocrystals over formal schemes. Let A be an I -adically complete Noetherian \mathbb{F}_p -algebra and let $\mathfrak{X} := \text{Spf}(A)$. The general goal is to study the category $\text{Isoc}(\mathfrak{X})$ when A admits

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a finite p -basis. Depending on the participants' expertise, this project can also lay the foundations for a stacky-theoretic approach to isocrystals over formal schemes, which would be useful for applications to Shimura varieties. The main references are [deJ95] and [Dri22].

Project B: Monodromy beyond constant slopes. Let (A, λ) be a principally polarised abelian variety over $\overline{\mathbb{F}}_p$. We want to study the monodromy group of the universal isocrystal over the deformation space of (A, λ) . In this project, the participants should have a good understanding of p -divisible groups and their deformation theory and should be familiar with algebraic groups. The main reference is [DvH22].

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